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MATHEMATICS 9

Measurement and Geometry

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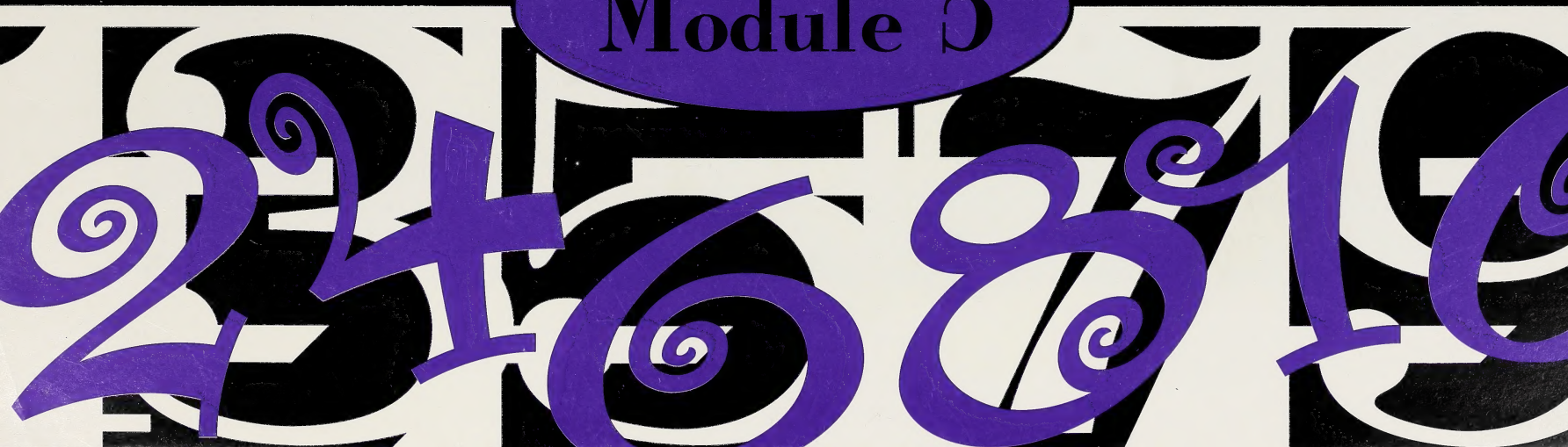
Module 5



Mathematics 9

Module 5

Measurement and Geometry

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Mathematics 9
Student Module Booklet
Module 5
Measurement and Geometry
Learning Technologies Branch
ISBN 0-7741-1390-1

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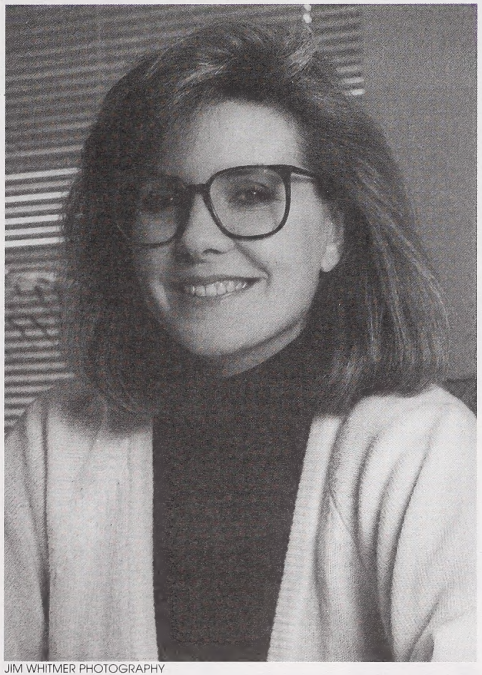
Welcome



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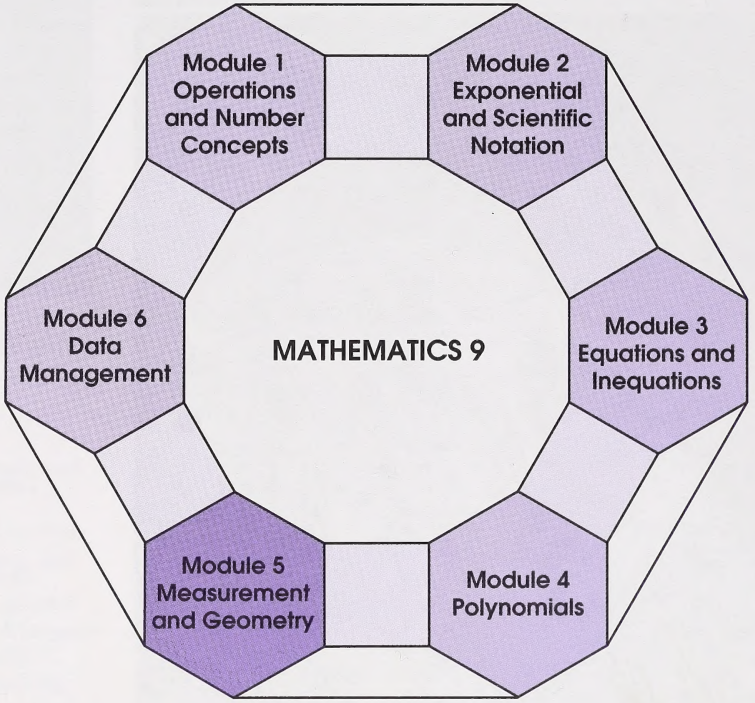
Welcome to Module 5. We hope you'll enjoy your study of Measurement and Geometry.

Mathematics 9 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Prepare for a problem that will provide a change of topic.

- Prepare for a challenging problem related to the topic of the activity.

- Use the Internet to explore a topic.

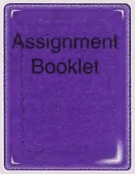
- Use computer software.

- Use a scientific calculator.

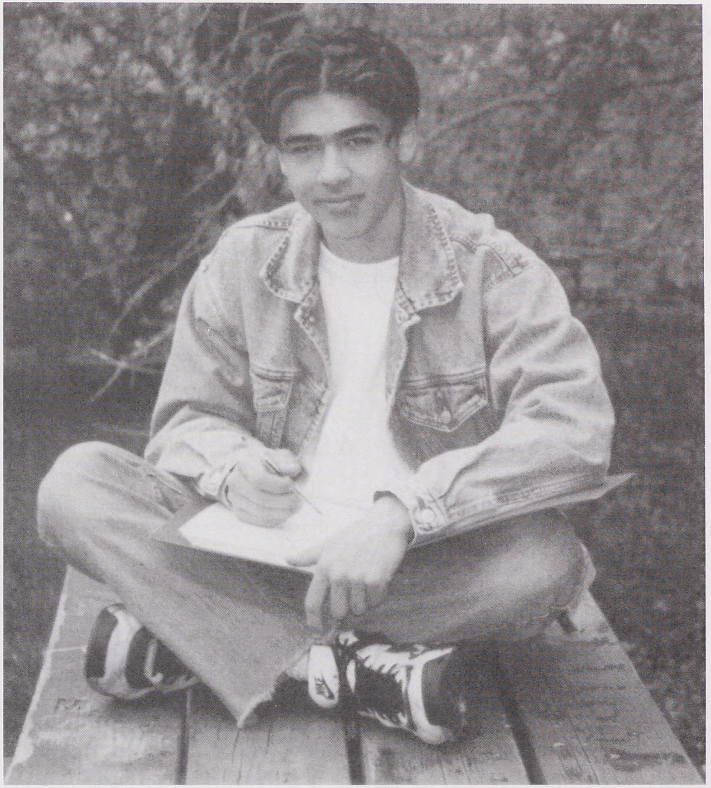
- View a videocassette.

- Pay close attention to important words or ideas.

- Use the suggested answers in the Appendix to correct activities.



- Answer the questions in the Assignment Booklet.



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There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

Technology

Today society is turning to **technology** more than ever before, and it is to your advantage to be able to effectively use technology when required.

Technology is the application of tools, materials, and processes to the solution of problems. More specifically, technology refers to devices and systems that are used in processing, transferring, storing, and communicating information through electronic media.

In Mathematics 9, along with the course materials, you will use a calculator, computer, and videocassette player as tools for learning and doing mathematics.

Calculators are helpful tools for solving problems and exploring patterns and relationships between numbers. Using a calculator will also save you time and help you develop your estimating skills. Therefore, you will be given numerous opportunities in each module to use a calculator.

Computers are useful for organizing and displaying data, or drawing figures. For this reason you will have the chance in many activities to work with popular computer applications such as spreadsheets and draw programs. You will also want to check out the many Internet connections in each module.

Videocassette players allow you to view video programs on key concepts that are difficult to explain in print. That is why video programs are cited in this course.

It is expected that all students will be able to view the video programs and use a calculator, and that most students will do the computer activities. However, if you are unable to access a computer, you may do the calculations using a calculator or draw figures and graphs by hand.

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Problem-Solving Skills

One of the exciting features of this course is that you will develop and improve your ability in problem solving. You will need these problem-solving skills many times in your lifetime. Since this course focuses on problem solving, it is important that you understand what a **problem** is.

A problem is a task for which the method of finding the answer (as well as the answer) is not immediately known.

Like any skill, the skill of problem solving must be developed. Problems may or may not involve computation (adding, subtracting, multiplying, and dividing). Some problems are realistic; others are puzzles.

You will have the opportunity in most activities to try a problem-solving challenge.

The icon is a cue that the problem will be related to the topic of the activity.

The icon is a cue that the problem will provide a change of topic.

The Four-Stage Process

There are four stages that can be used to solve any problem: understanding the problem, developing a plan, trying the plan, and looking back.

Understanding the Problem

In this stage you should expect to feel puzzled. There are various reasons for feeling this way.

- You may not know the meanings of all the words.
- You may not understand the situation in the problem.
- You may be confused by unnecessary information.

Once you understand the problem, you should think about the problem and make an estimate of what the answer should be. This will help you arrive at a reasonable answer.

Developing a Plan

This is where you should decide on the plan of action that you are going to take to solve the problem. You may consider the following strategies:

- changing your point of view
- using objects
- using diagrams
- making an organized list
- using Venn diagrams
- making a table
- guessing, checking, and revising
- acting out a problem
- working backwards
- simplifying a problem
- finding and applying a pattern
- using elimination
- using truth tables
- using an equation

Note: The Appendix in this module explains these strategies in detail. When you see a problem-solving icon in any module, you should turn to this Appendix and review the problem-solving strategies.

Trying the Plan

In this stage you should try the plan and see if it works.

Be sure to work carefully and record your progress. You are encouraged to use a calculator to help with your calculations.

Note: While trying the plan, you should monitor your progress in order to determine if your plan will lead to a solution. You may find that the plan will not produce a solution, in which case a new plan will have to be developed.

Looking Back

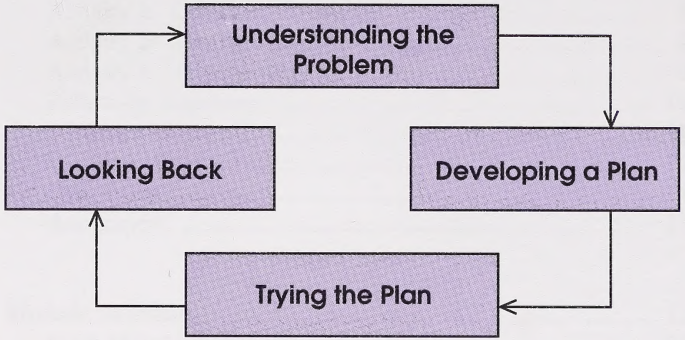
In this stage you should look back at the problem and compare your answer to the estimate you made in the first stage. Restate the problem using your answer.

Ask yourself these questions: "Did my plan work? Is my answer reasonable?"

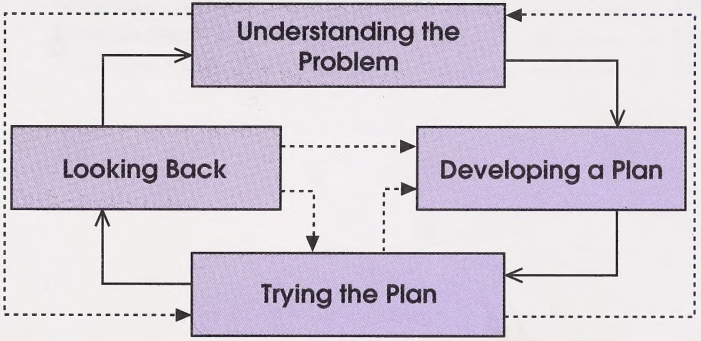
If you did not arrive at an answer, another strategy may work better. If your answer is unreasonable, you may have made errors while trying your plan.

Sequence of Stages

You usually approach a problem in the order outlined in the following diagram.



If you encounter difficulties in your original plan, or if you realize that another strategy will have better results, you may need to return to an earlier stage or use the stages in a different sequence.



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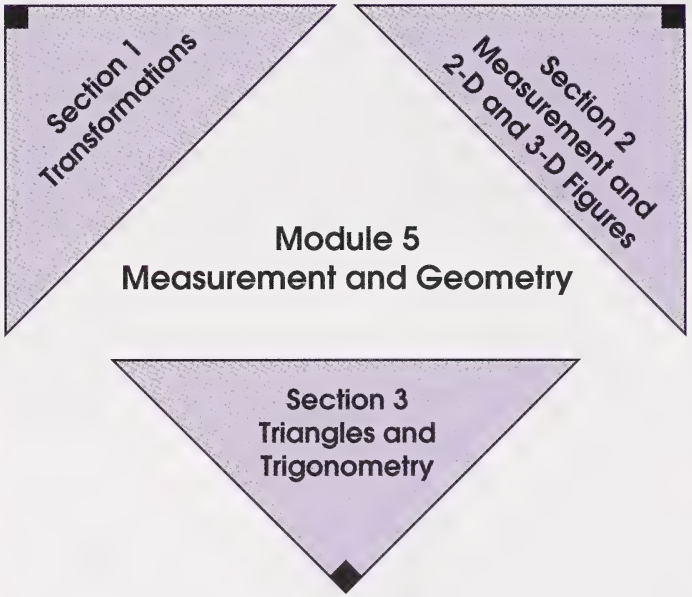
Module Overview

Did you perform some type of measurement today? Measurement in various forms is a vital part of everyday life, as is knowledge of certain aspects of geometry. Everything from the right amount of toothpaste, to the shortest distance to a friend's house, to the ability to draw in perspective all involve accurate measurement and a grasp of certain geometric concepts. You do many of these tasks without consciously thinking of measuring because they have become automatic to you through repetition. Other tasks you do may require thoughtful consideration and careful calculations.

In Section 1 of this module, you will extend your knowledge of transformations as you analyse translations, reflections, rotations, and dilatations.

In Section 2 you will analyse spatial problems involving points and geometric shapes. You will discover how to find the volume of pyramids and cones and how to relate these volumes to the volumes of prisms and cylinders.

Lastly, in Section 3 you will expand your knowledge of right triangles by examining the ratios of the lengths of the sides of a right triangle. These trigonometric ratios will help you to solve more problems involving right triangles.

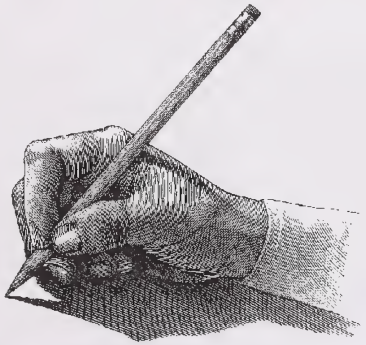


Evaluation

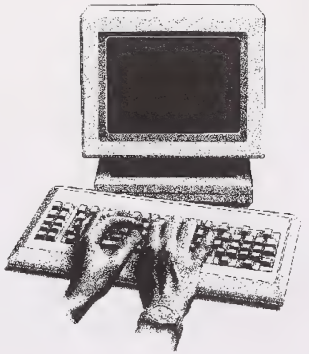
Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete three section assignments and a final module assignment. The mark distribution is as follows:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**25 marks**
Section 3 Assignment	**25 marks**
Final Module Assignment	**25 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently, but if you are having difficulties, you may review the appropriate section in this module booklet.



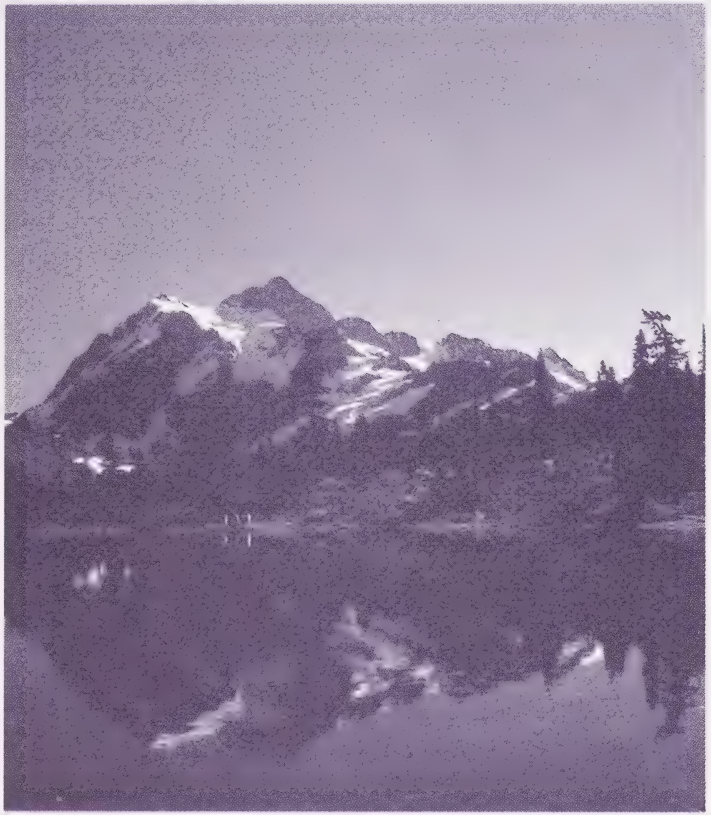
If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a supervised final test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Transformations



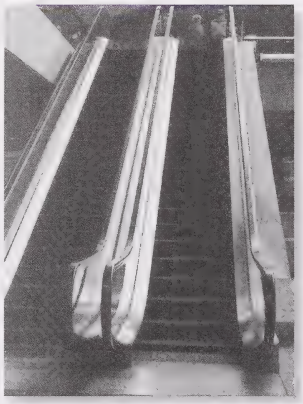
A mountain range reflected in a nearby lake is a form of transformation that provides a picturesque scene. There are numerous examples of transformations that appear in nature and in the world around you. For instance, transformations may appear in art and design, architecture, plants, geology, and engineering technology. Whenever a figure or object changes its shape, size, or position, it undergoes a transformation.

Transformations occur in mathematics whenever you slide (translate), reflect, rotate, or dilatate a figure. In previous mathematics courses you became familiar with translations, reflections, and rotations. Now, you are going to expand your knowledge of these transformations and examine other types of transformations.

In this section you will look at individual transformations (translations, reflections, rotations, and dilatations) of a figure on a coordinate grid. You will then look at combinations of transformations of a figure on a coordinate grid.

Activity 1: Translations

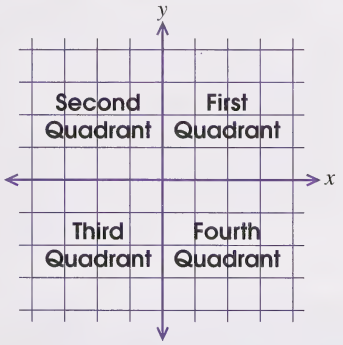
Have you gone up an escalator in a large department store or shopping mall? Did you know that your movement from the bottom of the escalator to the top (or vice versa) is an example of a **translation**?



A translation is a sliding motion with no turns or rotations.

Other examples of translations include a move on a chessboard and a person skiing down a hill. Repeated designs on floor coverings or wallpaper may also be examples of translations, provided there are no rotations in the pattern.

You can investigate translations by drawing a figure on a coordinate grid. You may recall from a previous mathematics course that a coordinate grid consists of a pair of intersecting axes that divide the grid into four quadrants (as shown in the diagram).

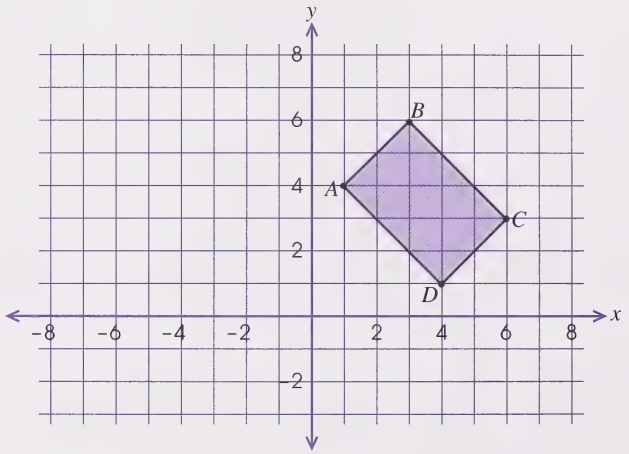


Points are located with respect to the intersection of the x- and y-axis and are written as ordered pairs with the x-coordinate first.

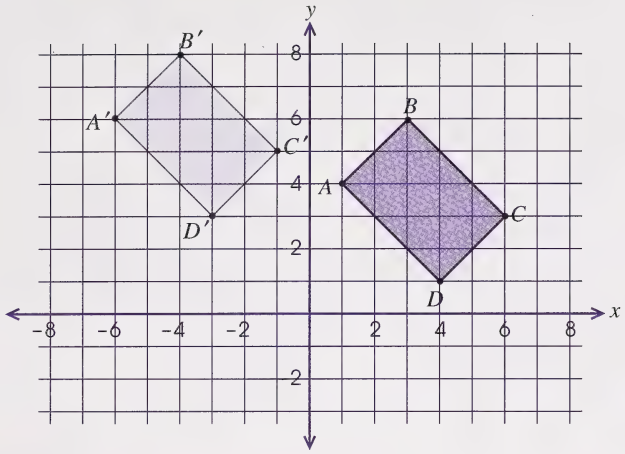
Use the following steps to translate a figure on a coordinate grid.

Step 1: Obtain a sheet of graph paper and draw a set of axes on it.

Step 2: Draw a figure such as a rectangle in the first quadrant. For convenience, be sure each vertex is on a whole-number coordinate. Label each vertex as shown.



Step 3: Move each vertex 7 units to the left and 2 units up, so that the entire rectangle is in the second quadrant. Label each vertex on the rectangle in the second quadrant.



The rectangle $A'B'C'D'$ is called the **image** of rectangle *ABCD*.

1. Measure the length of the sides and the size of the angles in both rectangles. What do you notice?

2. How is rectangle $A'B'C'D'$ different from rectangle *ABCD* with respect to the coordinate grid?

3. Draw arrows from each vertex of rectangle *ABCD* to the corresponding vertex of rectangle $A'B'C'D'$. What do you notice about the arrows you have drawn?

4. What does the answer to question 3 indicate about the corresponding sides of the original rectangle and those of its image?

5. If you could slide rectangle $A'B'C'D'$ onto rectangle *ABCD*, would it fit exactly?

6. Are rectangles *ABCD* and $A'B'C'D'$ congruent?

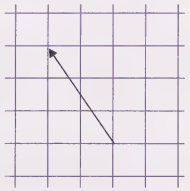
Congruent figures have exactly the same size and shape. The symbol ≅ is used to indicate that two figures are congruent.

Check your answers by turning to the Appendix.

Properties of a Translation

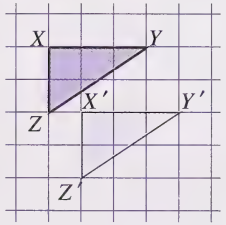
- Only the position of the original figure changes in a translation.
- The lines joining corresponding points of an original figure and its image are parallel.
- The corresponding sides of an original figure and its image are parallel.

You can describe a translation in several ways:

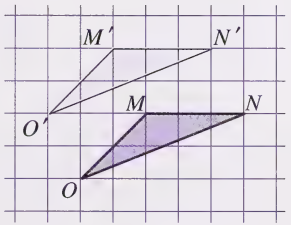
- in words
  e.g., Draw the image 7 units to the left and 2 units up.
- using an ordered pair
  e.g., $(4, -1)$ ← The figure is to be translated 4 units to the right and 1 unit down.
- using an arrow
  e.g., 
  ← The figure is to be translated 2 units to the left and 3 units up.

7. Describe each of the following translations in words, using ordered pairs, and using arrows.

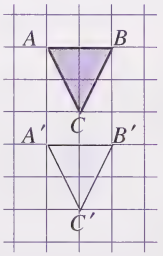
a.



b.

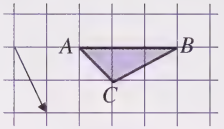


c.

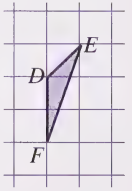


8. Copy the following figures on some graph paper, and complete the indicated translation.

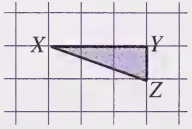
a.



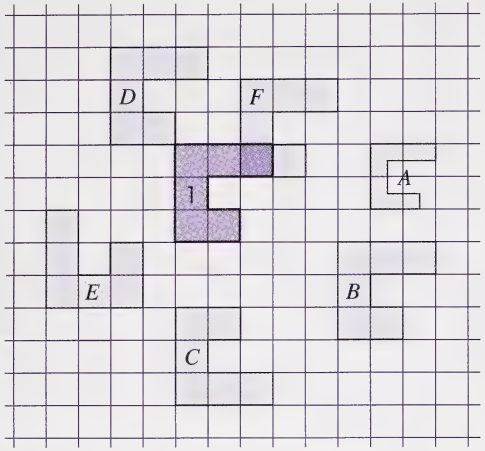
b. $(2, -2)$



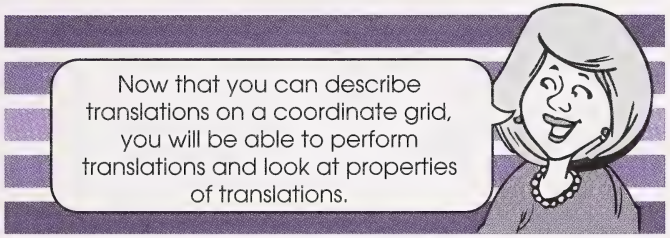
c. 1 unit to the left and 3 units down



9. Which of the following figures are translation images of figure 1? Describe each translation using ordered pairs.



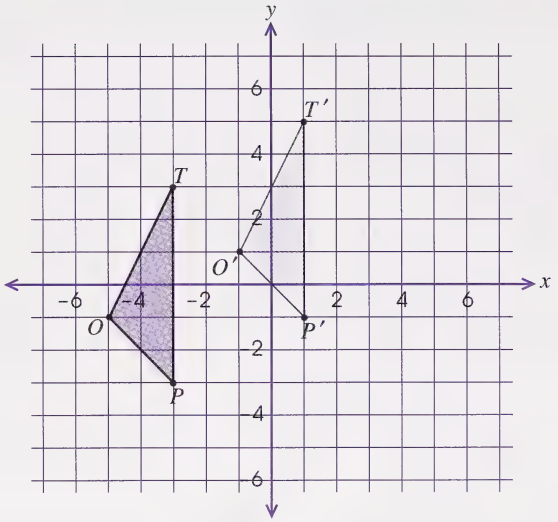
Check your answers by turning to the Appendix.



Example 1

Plot ΔTOP—with coordinates $T(-3,3)$, $O(-5,-1)$, and $P(-3,-3)$—on a coordinate grid; then translate each point 4 units to the right and 2 units up. Label the new image appropriately.

Solution



10. Study Example 1 carefully; then answer the following questions.

a. What are the coordinates of the vertices of the image $\Delta T'O'P'$?

b. Compare the lengths of the corresponding sides of ΔTOP and $\Delta T'O'P'$.

c. Compare the measures of the corresponding angles of ΔTOP and $\Delta T'O'P'$.

d. Are the two triangles congruent? Explain.

e. How are the x-coordinates of each vertex in $\Delta T'O'P'$ related to the x-coordinates of the corresponding vertices in ΔTOP?

f. How are the y-coordinates of the corresponding vertices related?

Check your answers by turning to the Appendix.

Example 2

Quadrilateral $LMOP$ with vertices $L(-5,-2)$, $M(-2,-3)$, $O(-3,-5)$, and $P(-6,-5)$ has been translated 8 units to the right and 6 units up. Find the coordinates, without graphing, of the image $L'M'O'P'$.

Solution

Add 8 units to each x-coordinate and 6 units to each y-coordinate.

$$L'(-5+8,-2+6)=L'(3,4)$$
$$M'(-2+8,-3+6)=M'(6,3)$$
$$O'(-3+8,-5+6)=O'(5,1)$$
$$P'(-6+8,-5+6)=P'(2,1)$$

The coordinates of each vertex of the image are $L'(3,4)$, $M'(6,3)$, $O'(5,1)$, and $P'(2,1)$.

11. Find the coordinates of the image of each point after a translation of 5 units to the right and 3 units up.

a. $P(-7,2)$ **b.** $W(6,4)$

c. $M(-2,-5)$ **d.** $N(0,9)$

12. Draw a triangle with vertices $X(-3,0)$, $Y(2,-2)$, and $Z(3,3)$. On the same grid, draw the image triangle for the following translations. **Note:** Use the notation $X''Y''Z''$ to label the triangle resulting from the translation in part b.

a. $(2,2)$

b. a translation of 3 units left and 3 units down

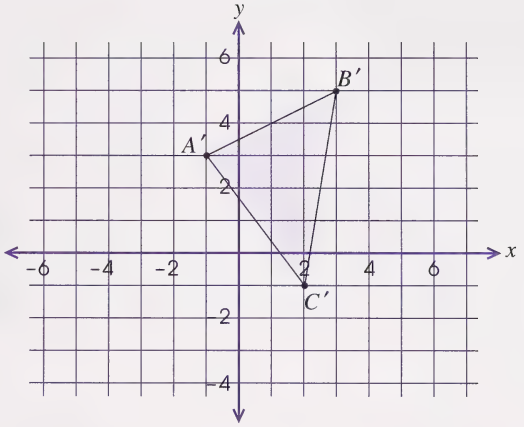
13. Are the sides of $\Delta X'Y'Z'$ in question 12 parallel to the corresponding sides of $\Delta X''Y''Z''$?

14. Write the translation for $\Delta X'Y'Z'$ to $\Delta X''Y''Z''$ in question 12.

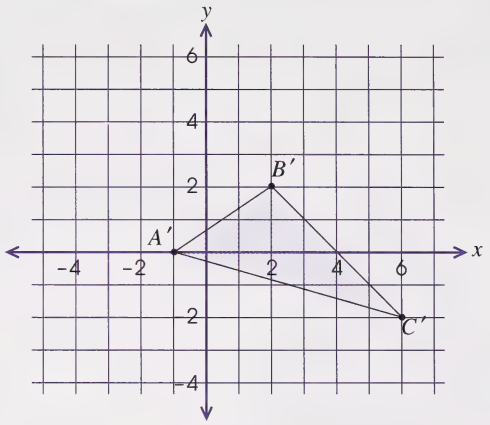
15. ΔPQR has vertices $P(5,-3)$, $Q(2,1)$, and $R(-3,-2)$. The translation image of ΔPQR has vertices $P'(9,-5)$, $Q'(6,-1)$, and $R'(1,-4)$. Identify the translation used to produce the image.

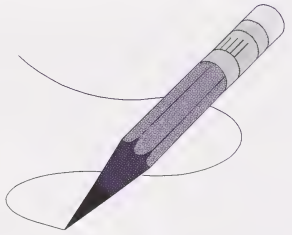
16. In each diagram, $\Delta A'B'C'$ is the image of ΔABC after the given translation. Find the coordinates of the vertices of ΔABC in each situation. (**Hint:** Perform the opposite translation to the one given to find the original figure.)

a. 4 units to the left and 2 units up

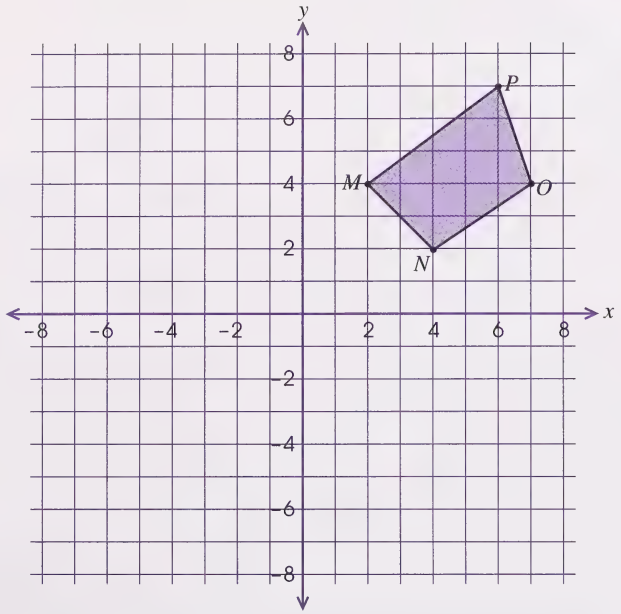


b. 3 units to the right and 1 unit down





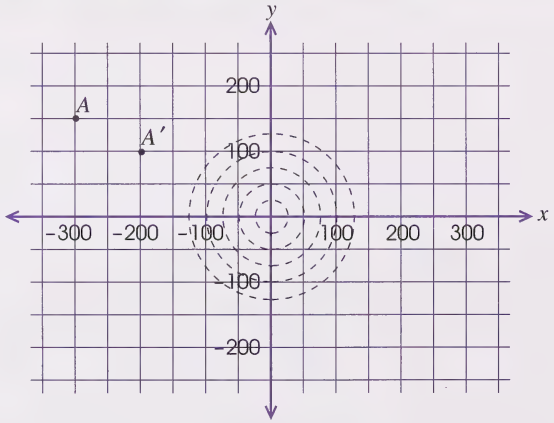
17. Quadrilateral *MNOP* lies in the first quadrant (as shown in the following graph).



a. Write a translation that would put the quadrilateral completely in the fourth quadrant.

b. Write a translation that would put the quadrilateral completely in the third quadrant.

18. At 11:35, an air traffic controller spots an airplane on the radar screen at coordinates $(-300, 150)$ (denoted by *A*). Five minutes later, the air traffic controller finds the plane at coordinates $(-200, 100)$ (denoted by A').

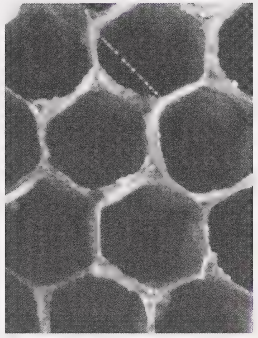


a. Assuming the airport is at coordinates $(0, 0)$, is the airplane on the correct heading to land at the airport? Explain.

b. If the airplane maintains its present speed for the next five minutes and then averages half that speed for the remainder of the flight, at what time should it touch down?

Check your answers by turning to the Appendix.

Did You Know?



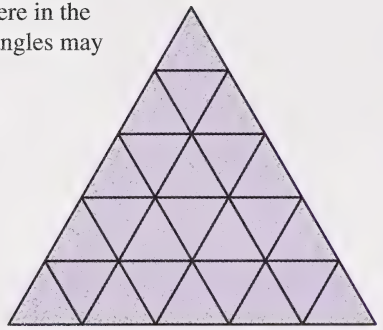
Repeated transformations can be used to create a tessellation. A tessellation is an arrangement of shapes that completely covers an area without overlapping or leaving gaps. The honeycomb in the photograph on the right is an excellent example of a tessellation involving translations. Tessellations are also found in some everyday objects such as fabric designs, wallpaper designs, linoleum designs, and patio block designs. Check various locations such as your home, your school, office buildings, or department stores for examples of tessellations involving translations.

If you have access to the Internet you may search for more information on tessellations. Enter the word "tessellations" in any available search engine.

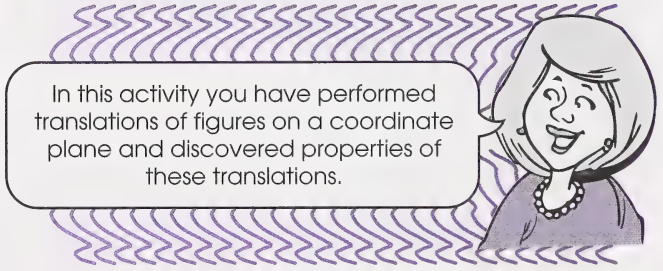
Now Try This

Use one of the problem-solving strategies to solve the following problem.

19. How many triangles are there in the diagram on the right? Triangles may overlap.



Check your answer by turning to the Appendix.



Activity 2: Reflections

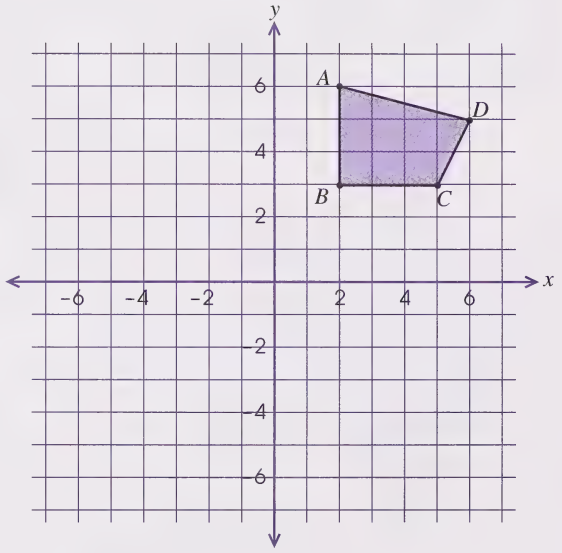
Look at your own image in a mirror, and raise your left hand. In what ways are you and your image alike? In what ways are you and your image different? Are you and your image congruent? Place your hand about 15 cm from the mirror. How far does the hand in the image appear to be from the mirror?

Your image in the mirror is called a **reflection**. Similarly, you can have a reflection of a figure in mathematics.

In mathematics, a reflection is a transformation that is represented by a flipping motion over a **mirror line** (or **reflection line**). The mirror line acts as a **line of symmetry** for the original figure and its image.

Use some graph paper to investigate the properties of reflections.

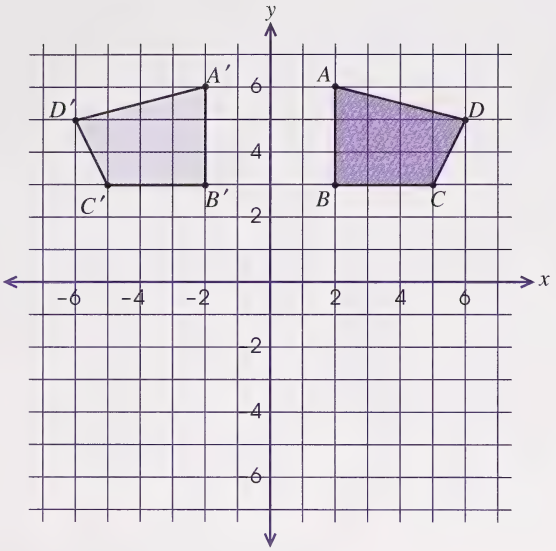
Step 1: Draw a quadrilateral exactly like the one shown in the following diagram.



Step 2: Fold the grid paper along the y-axis. Place the paper so you can see the letters of the drawing from the back of the folded side.

Step 3: Using a pen, make a heavy mark on each of the labelled points.

Step 4: Open up the grid paper and darken the points that show up on the left side of the vertical fold line. Join these points and label them $A'B'C'D'$. Your graph should look as follows:



1. Are the corresponding sides of quadrilaterals $ABCD$ and $A'B'C'D'$ the same length?

2. Are the measures of the corresponding angles of quadrilaterals $ABCD$ and $A'B'C'D'$ the same?

3. Trace quadrilateral $A'B'C'D'$ on some tracing paper. Can you slide quadrilateral $A'B'C'D'$ onto quadrilateral $ABCD$ so it covers it completely?

4. Is quadrilateral $A'B'C'D'$ a translation of quadrilateral $ABCD$? Explain.

5. How can you fit quadrilateral $A'B'C'D'$ onto quadrilateral $ABCD$ so it covers it completely?

Check your answers by turning to the Appendix.

Properties of a Reflection

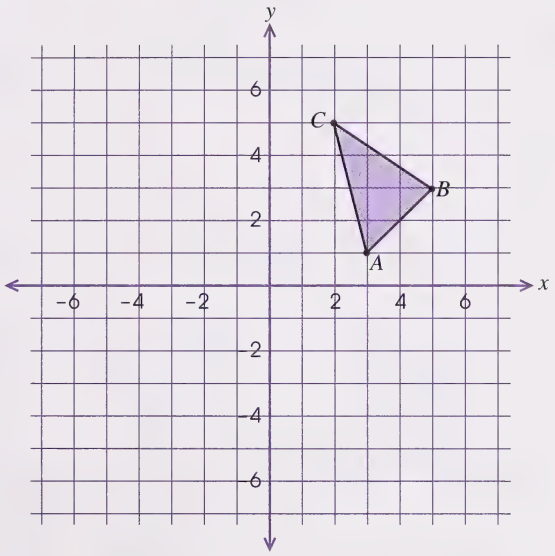
- Each point and its image are the same perpendicular distance from the line of reflection.
- The orientation of the image is opposite to the original figure.
- Lines joining corresponding points on the original figure and on the image are parallel.
- The reflection image is congruent to the original.

Example 1

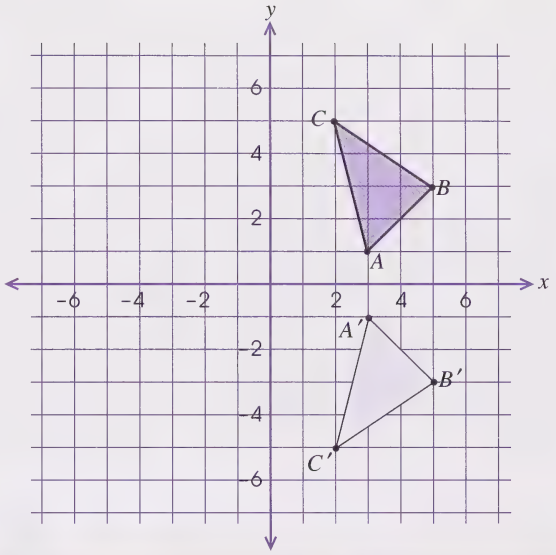
A triangle has vertices $A(3,1)$, $B(5,3)$, and $C(2,5)$. Draw the triangle and its image if the triangle is reflected about the x-axis.

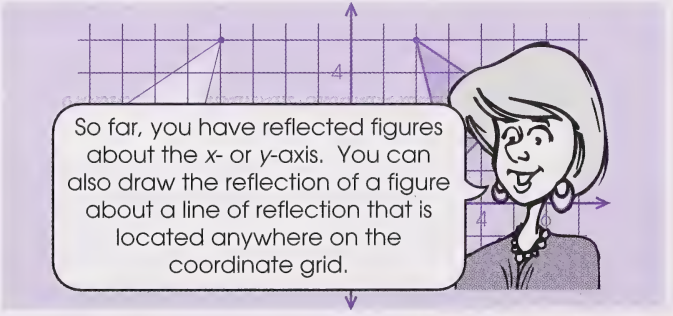
Solution

First, draw ΔABC on a coordinate grid.



Draw the reflection about the x-axis by placing the vertices of the image the same number of units below the x-axis as the vertices of the original are above. Join the vertices to complete the triangle, and label the image $A'B'C'$.



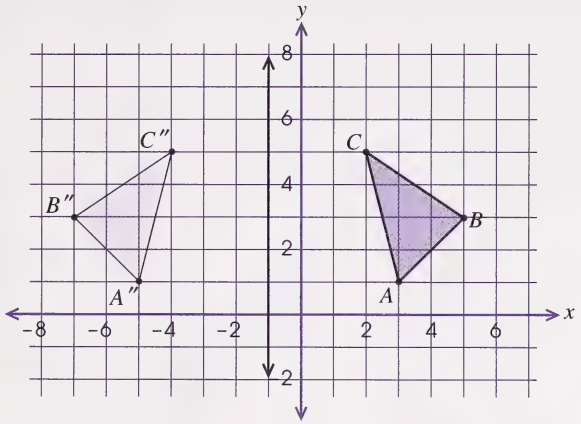


Example 2

Using the original triangle in Example 1, draw the image of the triangle if it is reflected about the line $x = -1$. **Note:** The line $x = -1$ is a vertical line 1 unit left of the y-axis.

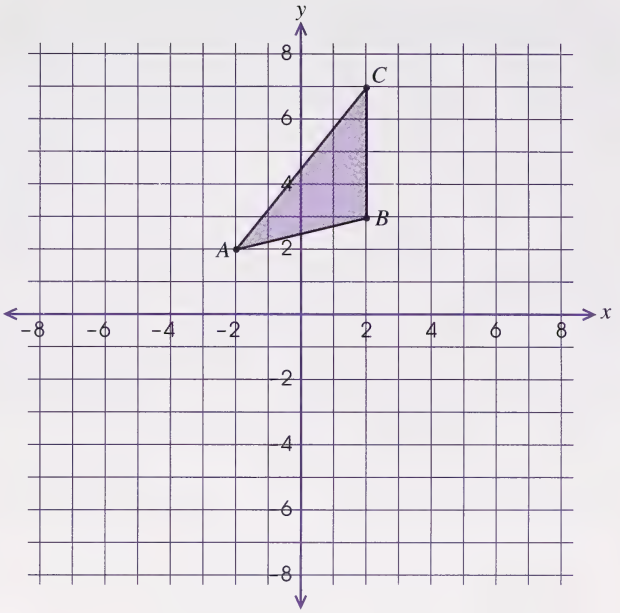
Solution

Draw the line $x = -1$; then locate the vertices of the image, which are the same number of units to the left of the line $x = -1$ as the vertices of the original are to the right of the line $x = -1$. Join the vertices to complete the triangle, and label the image $A''B''C''$.

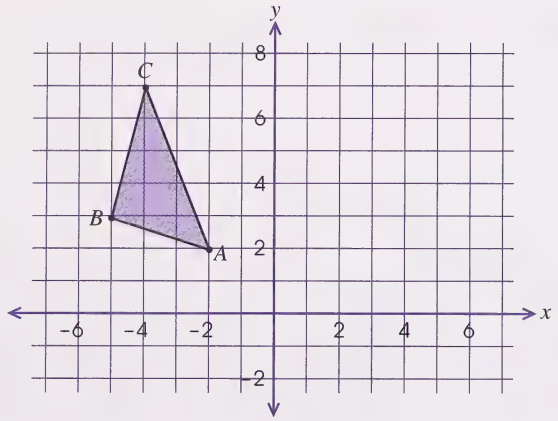


6. Copy the following figures on your own grid paper and complete the indicated reflection.

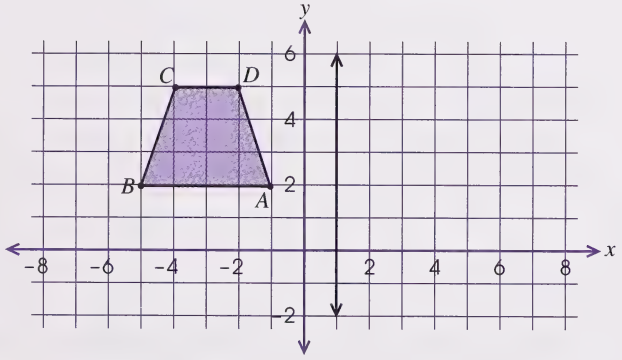
a. a reflection about the x-axis



b. a reflection about the y-axis

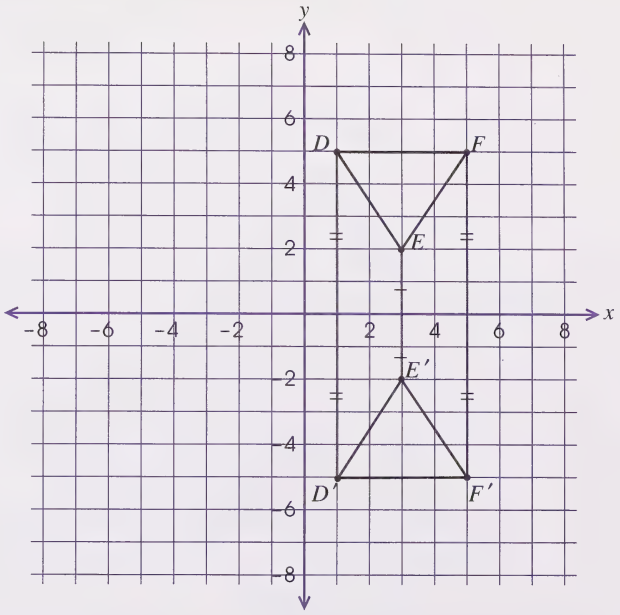


c. a reflection about the line $x = 1$

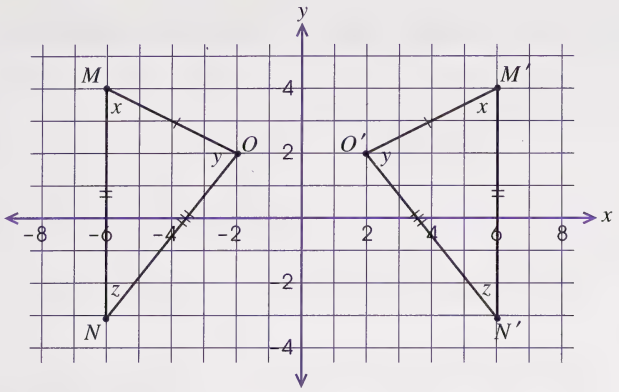


7. What property (or properties) of reflections is (are) illustrated in each of the following diagrams?

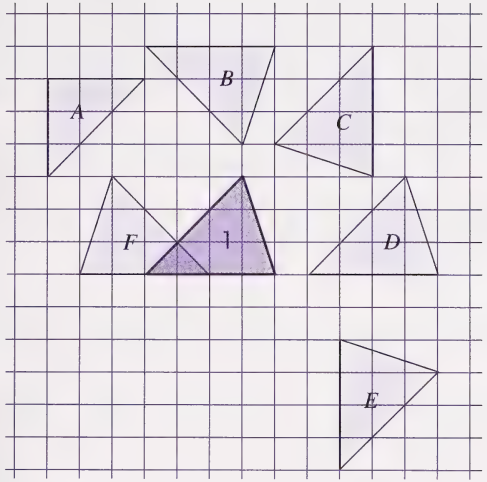
a.



b.



8. Which of the following figures are **not** reflection images of figure 1? Explain.



9. Copy and complete the following chart in your notebook. Write the coordinates of the reflection image for each point given. You may use a grid to plot each point and find the image point about each reflection line.

Point	Reflection Line	
	x-axis	*y*-axis
(2, 3)		
(−3, 4)		
(−1, −5)		
(0, 3)		
(4, 0)		
(4, −1)		

10. Use your answers to question 9 as a guide to complete the following mapping rules.

a. a reflection about the *x*-axis

$(x, y) \rightarrow (\quad, \quad)$

b. a reflection about the *y*-axis

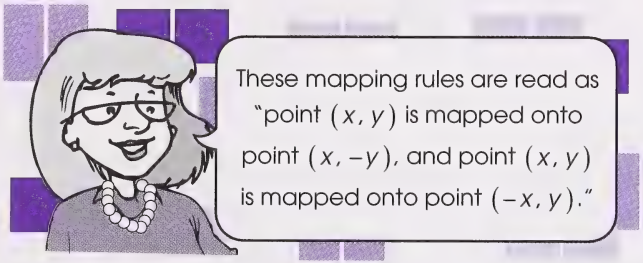
$(x, y) \rightarrow (\quad, \quad)$

11. Describe each mapping rule in question 10 in words.

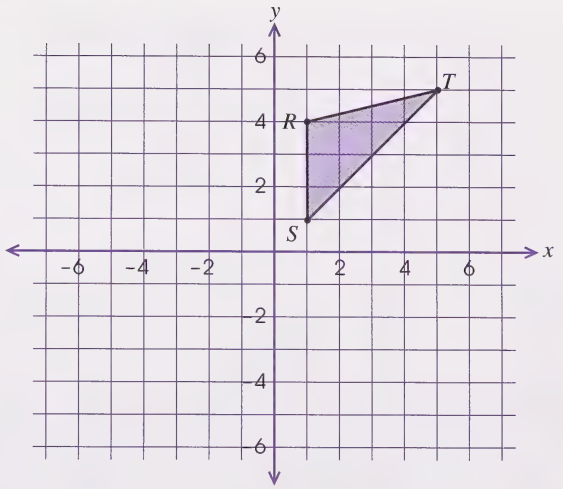
Check your answers by turning to the Appendix.

You now have mapping rules for reflections about both axes on a coordinate grid.

For a reflection about the x-axis, use the mapping rule $(x, y) \rightarrow (x, -y)$. For a reflection about the y-axis, use the mapping rule $(x, y) \rightarrow (-x, y)$.



12. Copy ΔRST onto a coordinate grid, and answer questions 12.a. to 12.e.



a. Draw the following images of ΔRST.

- an image, labelled $\Delta R'S'T'$, using the mapping rule $(x, y) \rightarrow (x, -y)$
- an image, labelled $\Delta R''S''T''$, using the mapping rule $(x, y) \rightarrow (-x, y)$
- an image, labelled $\Delta R'''S'''T'''$, using the mapping rule $(x, y) \rightarrow (-x, -y)$

b. Is $\Delta R'S'T'$ a reflection of ΔRST? Explain.

c. Is $\Delta R''S''T''$ a reflection of ΔRST? Explain.

d. Is $\Delta R'''S'''T'''$ a reflection of ΔRST? Explain.

e. What is $\Delta R'''S'''T'''$ a reflection of?

13. ΔMNO has vertices $M(-2,3)$, $N(2,4)$, and $O(1,-1)$.

a. What are the coordinates of its reflection image, $\Delta M'N'O'$, about the x-axis?

b. What are the coordinates of its reflection image, $\Delta M''N''O''$, about the y-axis?

14. a. When a point is reflected about the x-axis, which coordinate does not change?

b. When a point is reflected about the y-axis, which coordinate does not change?

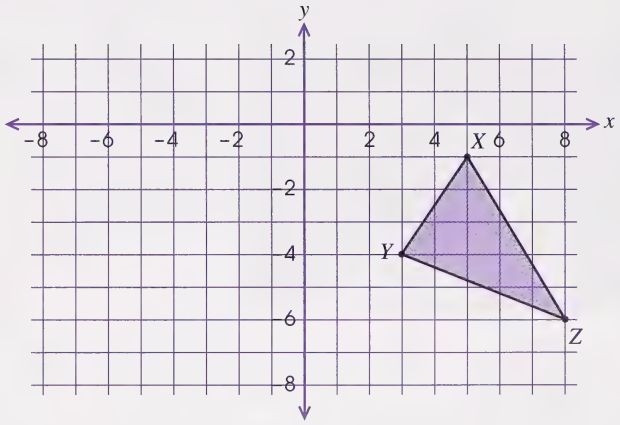
c. When neither coordinate changes after a reflection about the x-axis, where must the point be located?

d. When neither coordinate changes after a reflection about the y-axis, where must the point be located?

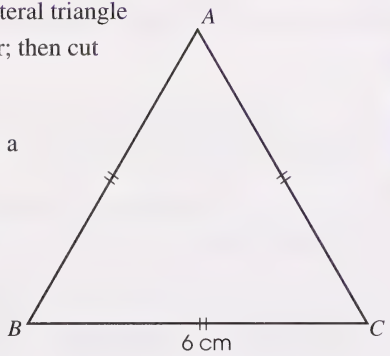
e. What point does not change after a reflection about either axis?

15. A triangle and its reflection have vertices $A(-5,1)$, $B(-2,2)$, and $C(-3,5)$ and $A'(-5,-3)$, $B'(-2,-4)$, and $C'(-3,-7)$ respectively. Draw ΔABC and $\Delta A'B'C'$ on a coordinate grid. Locate the line of reflection and write its equation.

16. On the coordinate grid given, locate the line of reflection such that the image of ΔXYZ will be located entirely in the third quadrant. Then draw the image $\Delta X'Y'Z'$. Restrict the value for the equation of the line of reflection to the set of integers.



17. Copy the following equilateral triangle (ΔABC) onto some paper; then cut it out.



a. Fold the paper in such a manner that the three vertices—A, B, and C—all meet together in the middle of the triangle. What is the shape formed?

b. What transformation did you use to construct this shape?

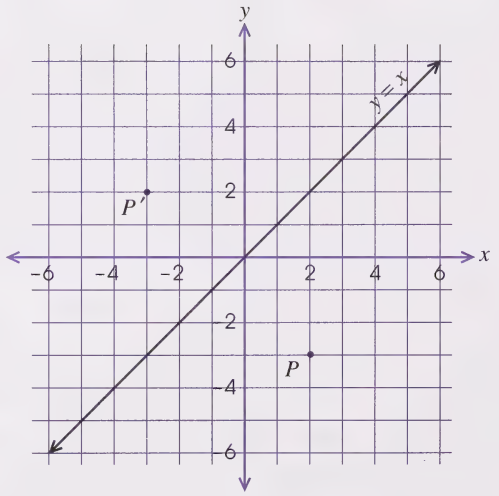
18. Kyle and Meagan are playing 8-ball (a billiards game). It is now Kyle's turn. He only needs to sink the 8-ball to win the game; but Meagan has blocked the direct path between the cue ball and the 8-ball with two of her own balls. So, Kyle must bank (bounce) the cue ball off the cushion (inside edge of the table) to hit the 8-ball. Explain, using a diagram and in words, how he has to aim the cue ball in order to hit the 8-ball.

Check your answers by turning to the Appendix.

Now Try This

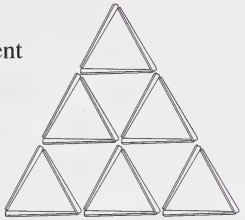
Solve the following challenge question.

19. P is reflected about the line $y = x$ as shown on the following grid. Locate at least two more points and their image points on the grid. Write the mapping rule you would use to find the reflection image about the line of reflection, $y = x$.



Use one of the problem-solving strategies to solve the following problems.

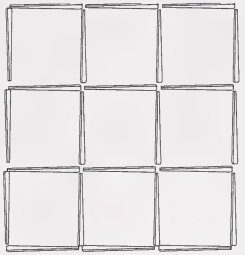
20. Obtain 18 toothpicks and arrange them according to the diagram.



a. Which three toothpicks should you remove in order to leave six congruent triangles?

b. Which six toothpicks should you remove in order to leave six congruent triangles?

21. Obtain 24 toothpicks and arrange them according to the diagram.



Which six toothpicks should you remove in order to leave four squares?

Check your answers by turning to the Appendix.

In this activity you discovered how to express the various properties of reflections and how to draw reflection images of figures on a grid using coordinates and mapping rules.

Activity 3: Rotations

Many carnival rides use rotation to provide thrill and excitement. The giant swing is one example. What happens to the position of your body, relative to the centre post, as the swing rotates?

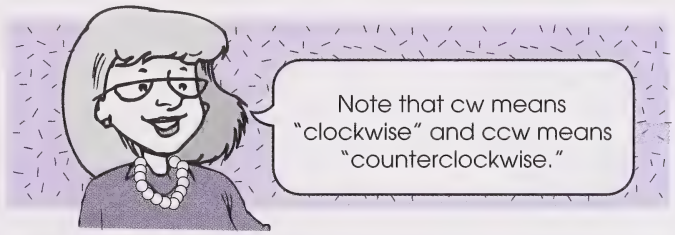
A merry-go-round, a pendulum on a clock, and the wheels on a bicycle are other examples of rotations.

In mathematics, a rotation occurs when you rotate a shape or figure about a fixed point. In previous mathematics courses you studied how a figure can be rotated about a point without changing its shape or size. Now you can apply this concept to the following activities involving rotations on a coordinate grid.

On a coordinate grid, the origin $(0, 0)$ is usually the **turn centre**. Points and their images will always be the same distance from the origin under these circumstances.

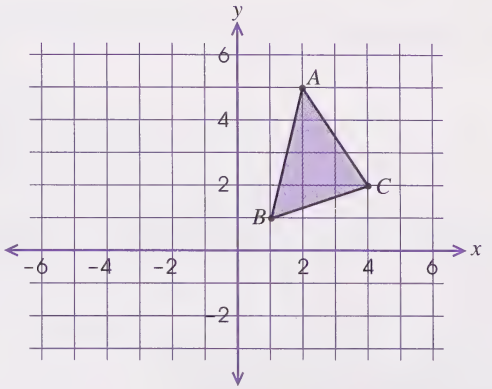
The turn centre will be labelled with a turn arrow indicating the amount of rotation. The turn arrows you may encounter are as follows:

90° cw	90° ccw
180° cw	180° ccw
270° cw	270° ccw



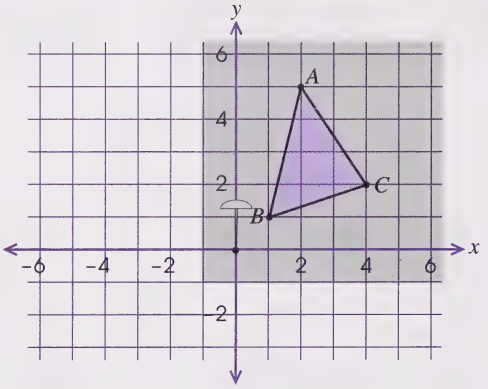
Perform the following steps to find the rotation images of ΔABC rotated 90° clockwise $\left(\frac{1}{4} \text{ turn cw}\right)$, 180° clockwise $\left(\frac{1}{2} \text{ turn cw}\right)$, and 270° clockwise $\left(\frac{3}{4} \text{ turn cw}\right)$ about the origin.

Step 1: Draw ΔABC with vertices $A(2, 5)$, $B(1, 1)$, and $C(4, 2)$ on a coordinate grid.



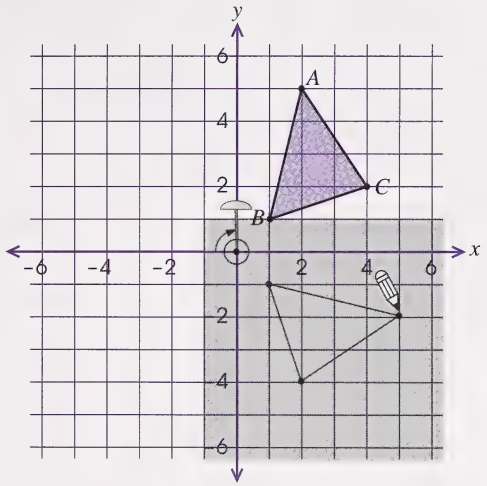
Step 2: Fasten the grid on a piece of cardboard using tape or glue.

Step 3: Lay tracing paper over the grid. Fix the tracing paper over the origin of the grid using a straight pin, compass point, or thumb tack.



Step 4: Using a pencil, mark the tip of the y-axis and points A, B, and C on the tracing paper.

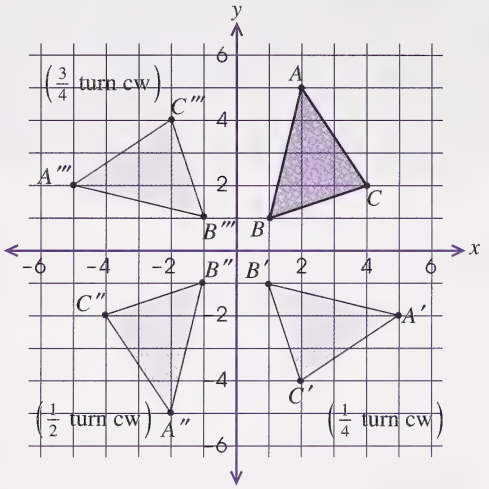
Step 5: Rotate the tracing paper 90° clockwise $\left(\frac{1}{4}\text{ turn cw}\right)$. (The mark from the tip of the y-axis should now be at the tip of the x-axis.) Make heavy marks at points A, B, and C on the tracing paper so that the marks go through to the grid. Connect the points on the grid and label them A', B', and C'.



Step 6: Rotate the tracing paper 180° clockwise $\left(\frac{1}{2}\text{ turn cw}\right)$ from its original position. (The mark from the tip of the y-axis should be at the tip of the $-y$-axis.) Mark the points in the same way as in Step 5. Connect the points on the grid and label them A'', B'', and C''.

Step 7: Rotate the tracing paper 270° clockwise $\left(\frac{3}{4}\text{ turn cw}\right)$ from its original position. (The mark from the tip of the y-axis should be at the tip of the $-x$-axis.) Mark the points in the same way as in Step 5. Connect the points on the grid and label them A''', B''', and C'''.

The completed grid should look as follows:



1. Complete the coordinates for each rotation by filling in the brackets.

$\frac{1}{4}$ turn cw

$\Delta ABC \rightarrow \Delta A'B'C'$

$A(2,5) \rightarrow A'(\quad, -2)$
$B(1,1) \rightarrow B'(1, \quad)$
$C(4,2) \rightarrow C'(\quad, \quad)$

$\frac{1}{2}$ turn cw

$\Delta ABC \rightarrow \Delta A''B''C''$

$A(2,5) \rightarrow A''(-2, \quad)$
$B(1,1) \rightarrow B''(\quad, \quad)$
$C(4,2) \rightarrow C''(\quad, -2)$

$\frac{3}{4}$ turn cw

$\Delta ABC \rightarrow \Delta A'''B'''C'''$

$A(2,5) \rightarrow A'''(\quad, \quad)$
$B(1,1) \rightarrow B'''(-1, \quad)$
$C(4,2) \rightarrow C'''(\quad, 4)$

2. Use the information from the preceding graph and question 1 to answer the following questions.

a. Are the image and the original figure congruent?

b. Does the orientation of the figure change?

c. How does each point and its image relate to the turn centre?

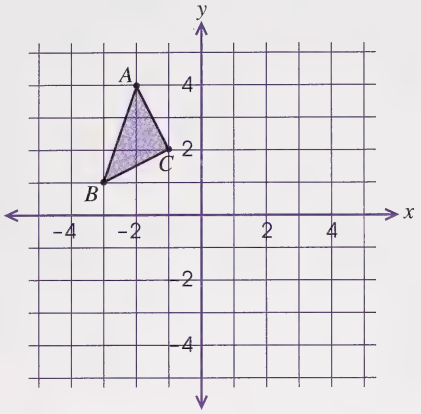
Check your answers by turning to the Appendix.

Properties of a Rotation

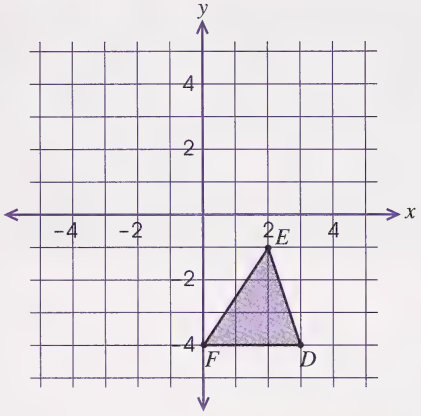
- The image and the original figure are congruent.
- The orientation of the image is different than that of the original figure.
- A point and its image are the same distance from the turn centre.

3. Copy each figure that follows onto a coordinate grid; then draw the indicated rotation image about $(0, 0)$.

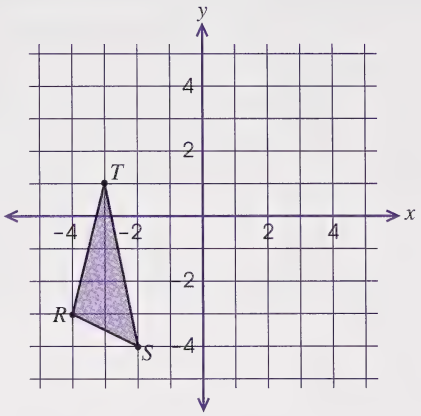
a. 270° cw



b. 180° cw



c. 90° cw



4. Use your answers to question 3 to write the coordinates of the vertices for each figure and its image.

5. Use your answers to questions 1 and 4 to complete the following mapping rules for rotations.

$\frac{1}{4}$ turn cw: $(x, y) \rightarrow (\quad , \quad)$

$\frac{1}{2}$ turn cw: $(x, y) \rightarrow (\quad , \quad)$

$\frac{3}{4}$ turn cw: $(x, y) \rightarrow (\quad , \quad)$

Check your answers by turning to the Appendix.

You now have mapping rules for rotations about the origin $(0,0)$.

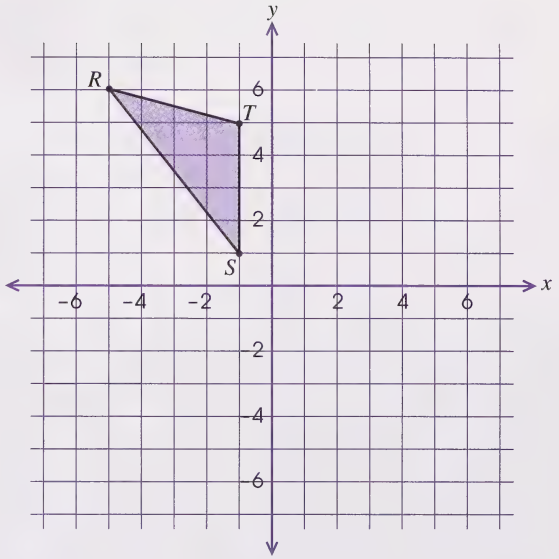
When drawing rotations on a coordinate grid using $(0,0)$ as the turn centre, use the following mapping rules.

Amount of Turn	Rotation Rule
$\frac{1}{4}$ turn cw	$(x, y) \rightarrow (y, -x)$
$\frac{1}{2}$ turn cw	$(x, y) \rightarrow (-x, -y)$
$\frac{3}{4}$ turn cw	$(x, y) \rightarrow (-y, x)$

The following turns are equivalent:

- $\frac{1}{4}$ turn clockwise (cw) is equal to $\frac{3}{4}$ turn counterclockwise (ccw).
- $\frac{1}{2}$ turn clockwise (cw) is equal to $\frac{1}{2}$ turn counterclockwise (ccw).
- $\frac{3}{4}$ turn clockwise (cw) is equal to $\frac{1}{4}$ turn counterclockwise (ccw).

6. Draw ΔRST on a coordinate grid, and answer the following questions.



a. Use the mapping rule $(x, y) \rightarrow (-x, -y)$ to draw the image of ΔRST. Label the image $\Delta R'S'T'$. Is $\Delta R'S'T'$ a rotation image of ΔRST? Explain.

b. Use the mapping rule $(x, y) \rightarrow (y, -x)$ to draw the image of ΔRST. Label the image $\Delta R''S''T''$. Is $\Delta R''S''T''$ a rotation image of ΔRST? Explain.

7. ΔMNO has vertices $M(-2,3)$, $N(1,-1)$, and $O(2,4)$. Use the appropriate mapping rules or a coordinate grid to answer the following questions.

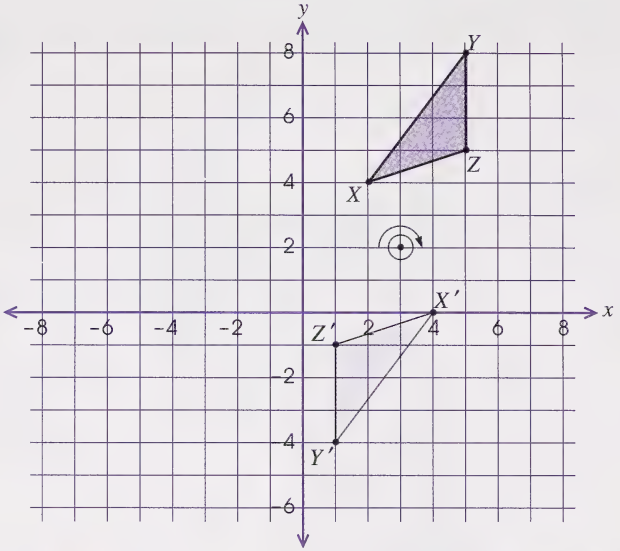
a. What are the coordinates of the image, $\Delta M'N'O'$, after rotating ΔMNO 90° clockwise about $(0,0)$?

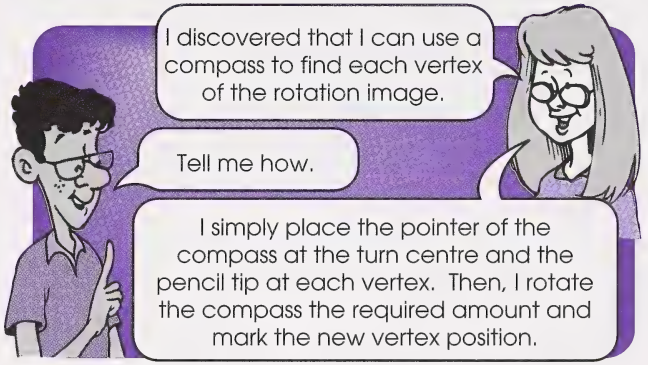
b. What are the coordinates of the image, $\Delta M''N''O''$, after rotating ΔMNO 180° clockwise about $(0,0)$?

c. What are the coordinates of the image, $\Delta M'''N'''O'''$, after rotating ΔMNO 270° clockwise about $(0,0)$?

Check your answers by turning to the Appendix.

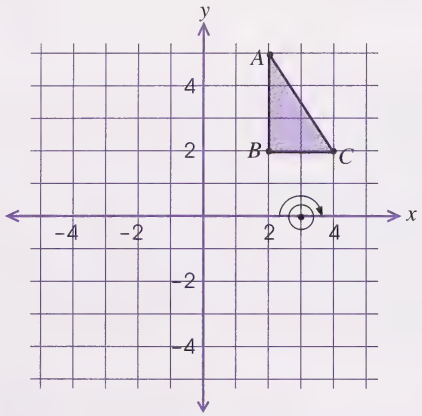
The origin, $(0,0)$, does not have to always be the rotation centre. For example, the turn centre could be the point $(3,2)$ as shown on the following grid. The image is a 180° cw rotation of the original figure relative to that point.



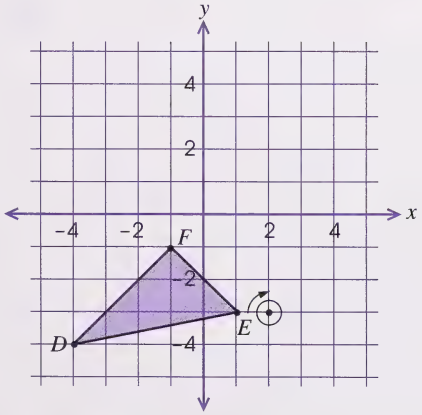


8. Draw the following diagrams on a coordinate grid, and draw the indicated rotation image of each figure about the given turn centre. Use tracing paper or use a compass to find each vertex.

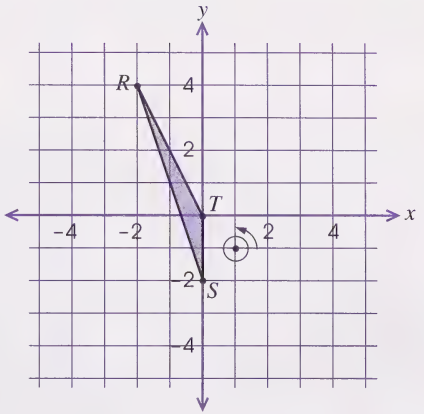
a.

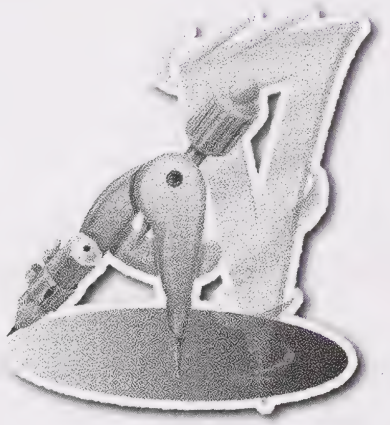


b.



c.





9. Copy and complete the following chart in your notebook. Find the coordinates of the image for each indicated rotation. You may use a coordinate grid or mapping rules.

Point	Rotation		
	90° cw	180° cw	270° cw
(0, 4)			
(−3, 0)			
(−2, −3)			
(3, 5)			
(1, −4)			

Check your answers by turning to the Appendix.

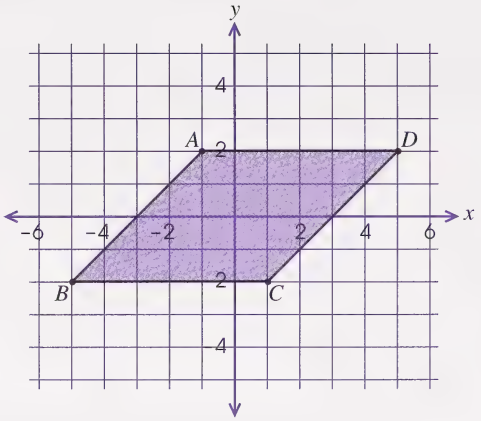
You have already used some of the knowledge you have gained about rotations to help you solve some problems. Now you will look at some different applications.

Example 1

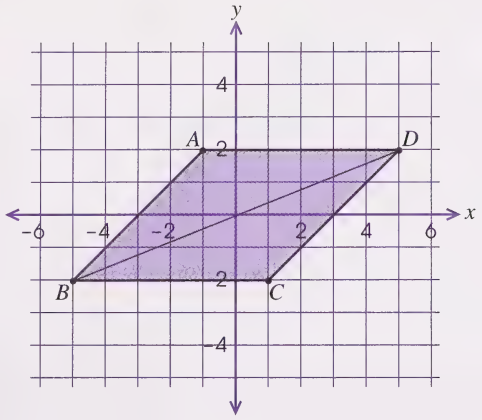
Show that the diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Solution

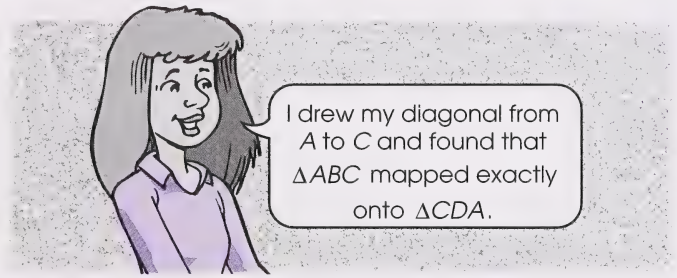
Draw a parallelogram with vertices $A(-1, 2)$, $B(-5, -2)$, $C(1, -2)$, and $D(5, 2)$ on a coordinate grid.



Draw the diagonal BD and rotate ΔABD a half turn about the origin.

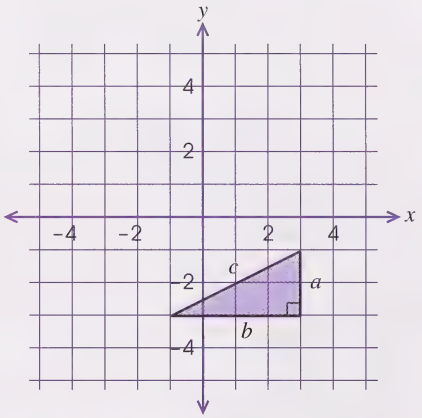


ΔABD maps exactly onto ΔCDB. Therefore, $\Delta ABD \cong \Delta CDB$.



10. Use rotations to show that the diagonals of a square divide the square into four congruent triangles.

11. **a.** Draw the following figure on a coordinate grid; then draw its $\frac{1}{4}$-turn image, its $\frac{1}{2}$-turn image, and its $\frac{3}{4}$-turn image about the origin.



Label the sides of each image like the original.

b. What shape is the outside figure?

c. What shape is the inside figure?

d. What is the area of the outside figure in square units?

e. What is the area of the inside figure in square units?

f. Show that the area of the outside square is equal to the area of the inside square plus the area of the four triangles.

12. Many corporations use logos that are designed using **rotational symmetry**. That is, the design will map onto itself as it is rotated. The recycle symbol shown here is one example.

a. Find at least two such logos.

b. Design a logo of your own for a mock corporation.

Check your answers by turning to the Appendix.

Now Try This

Solve the following challenge problems.

Many capital letters can be reflected and/or rotated and still read the same. Some examples are as follows:

A O D E M N S T W Y Z

13. Determine how each word can be transformed (reflected about a vertical line, reflected about a horizontal line, and/or rotated $\frac{1}{2}$ turn) and still read the same.

a. AIM **b.** OXO **c.** NOON

d. DEED **e.** HOAX

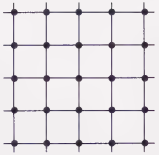
14. Find two words that fit each kind of transformation described in question 13. Do not use any words used before.

Use one of the problem-solving strategies to solve the following problem.

15. Copy the following dot patterns onto your own graph paper and answer questions 14.a., 14.b., and 14.c. You must connect the appropriate dots to get the figure that satisfies all the requirements.

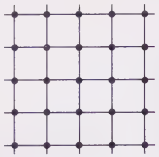
a. Connect the dots to draw a triangle with

- 10 dots outside the triangle
- 3 dots inside the triangle
- an area of 8 square units



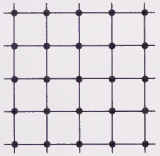
b. Connect the dots to draw a square with

- 12 dots outside the square
- 5 dots inside the square
- an area of 8 square units

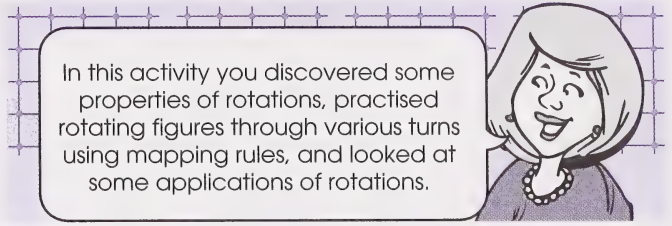


c. Connect the dots to draw a square with

- 9 dots outside the square
- 4 dots inside the square
- an area of 9 square units

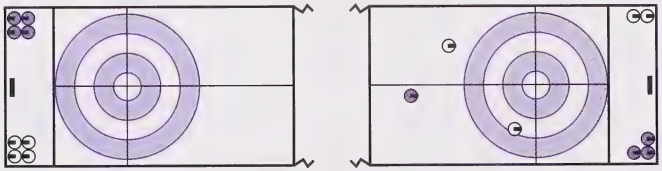


Check your answers by turning to the Appendix.



Activity 4: Applying Two or More Transformations in Succession

Many times you will see combinations of different transformations used. Curling is an activity that displays more than one type of transformation.



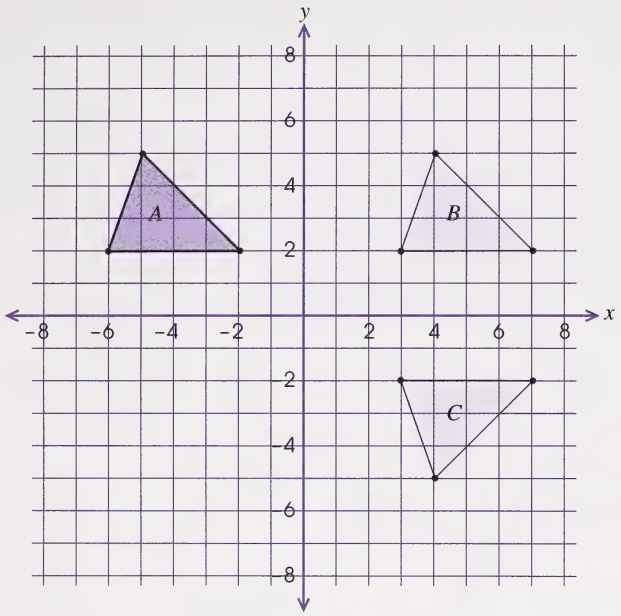
The rocks rotate as they slide down the ice. When a takeout is made (a rock is hit by a thrown stone), a translation of the stationary rock results. Each half of the curling sheet is a reflection of the other.

In mathematics several transformations can be combined in succession, resulting in a movement over several quadrants on the coordinate plane.

The following examples show combinations of transformations.

Example 1

What transformations are used to map figure A onto figure C through figure B?

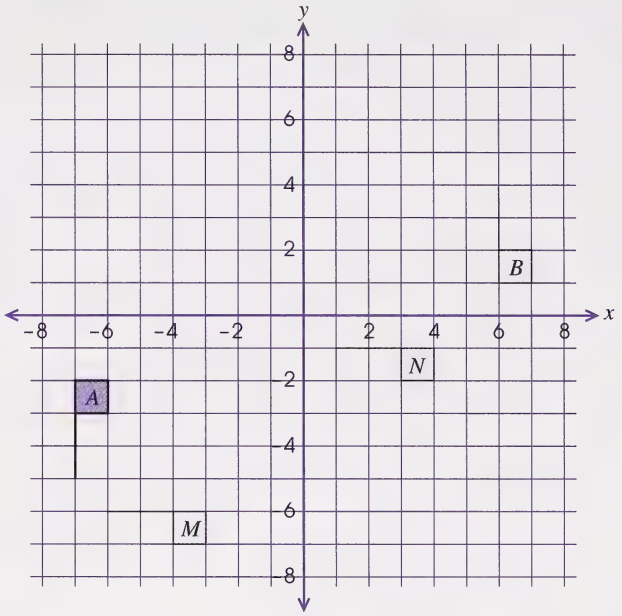


Solution

A translation of 9 units to the right moves figure A onto figure B. Then, a reflection about the x-axis maps figure B onto figure C.

Example 2

What transformations are used to move flag A onto flag B through flags M and N?

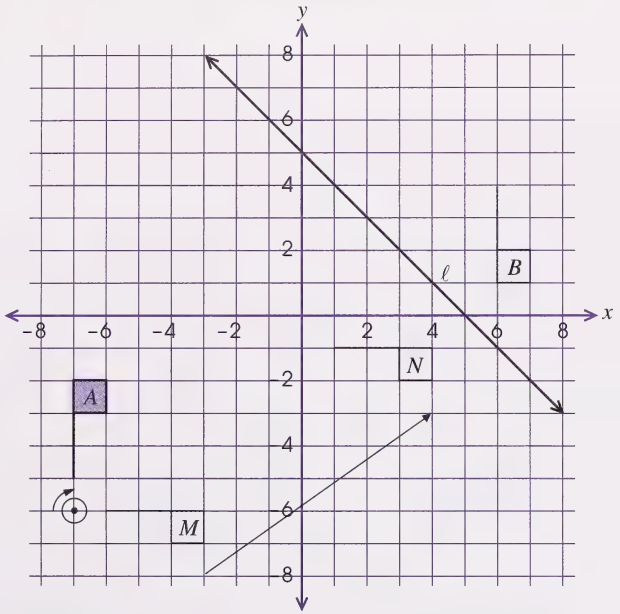


Solution

The following is one possible solution:

- A $\frac{1}{4}$ turn clockwise about point $(-7, -6)$ maps flag A onto flag M.

- A translation of 7 units to the right and 5 units up maps flag M onto flag N.
- A reflection about line ℓ maps flag N onto flag B.



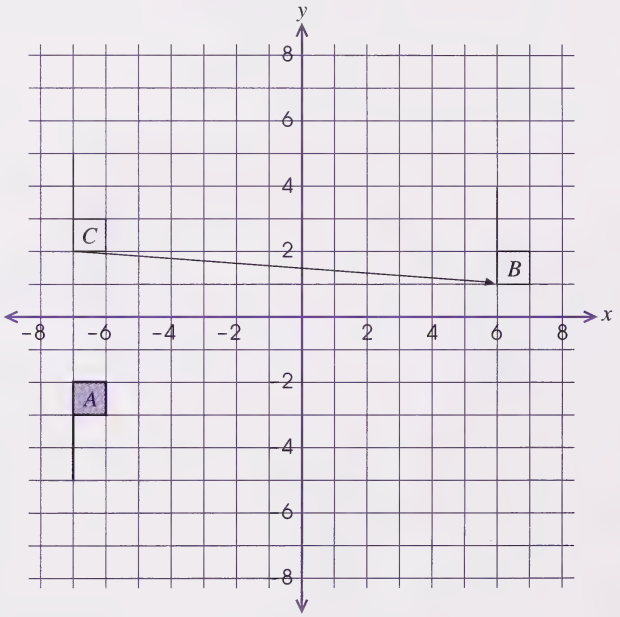
Many times the sequence of transformations used to map one figure onto another can be reduced in number, or the order of transformations can be changed without affecting the final result. In Example 1, triangle A could be translated into the third quadrant and then flipped to fit on triangle C.

Example 3

Map flag A from Example 2 onto flag B using only a reflection and a translation.

Solution

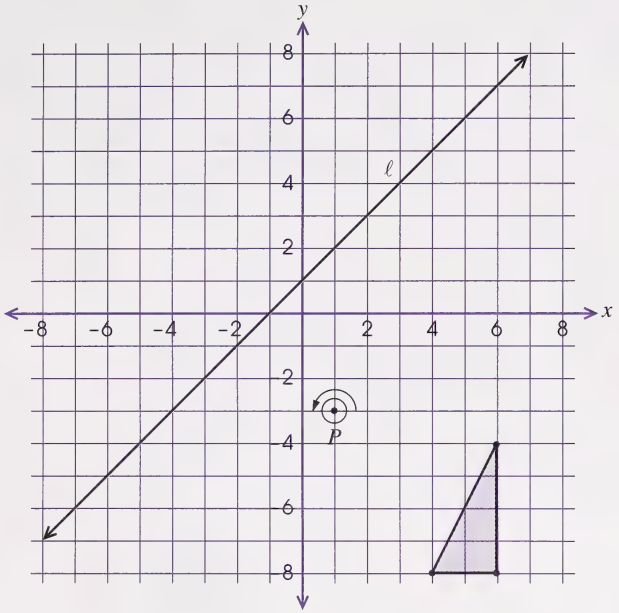
Reflect flag A about the x-axis; then translate the image 13 units to the right and 1 unit down (as shown in the following diagram).



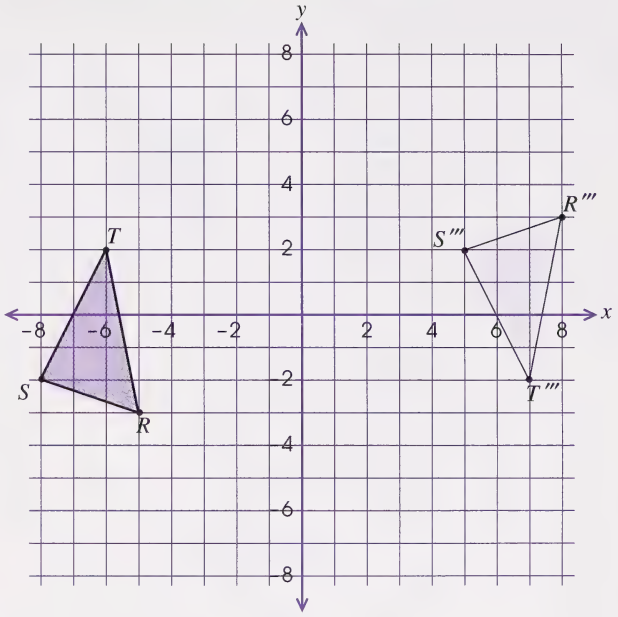
1. Copy the figure given onto a coordinate grid; then perform the indicated transformations in the following order:

- Translate 2 units to the left and 3 units up.
- Rotate 180° ccw about P.
- Reflect about line ℓ.

Show the original and the three images on the same grid.

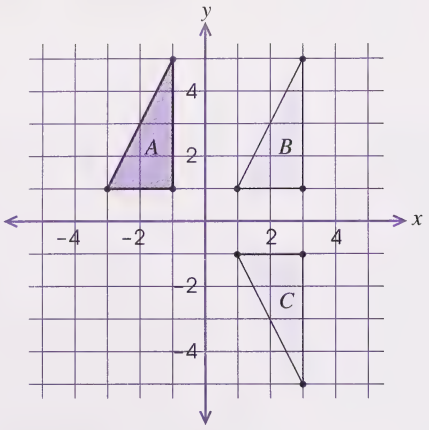


2. Describe a sequence of transformations that would map ΔRST onto $\Delta R'''S'''T'''$. You must use one translation, one reflection, and one rotation (not necessarily in that order). Show all necessary turn centres, reflection lines, and slide arrows; and describe the sequence you used in writing.

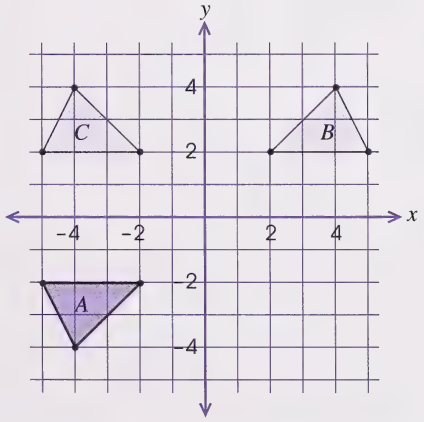


3. For each of the following, write the mapping rule that transforms figure A onto figure B and figure B onto figure C.

a.



b.

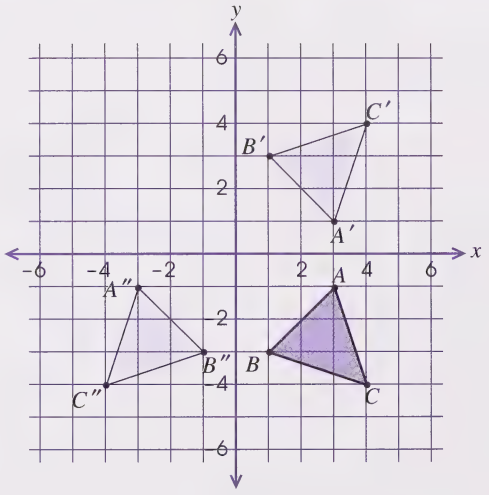


4. Write the single mapping rule that transforms figure A onto figure C in question 3.b.

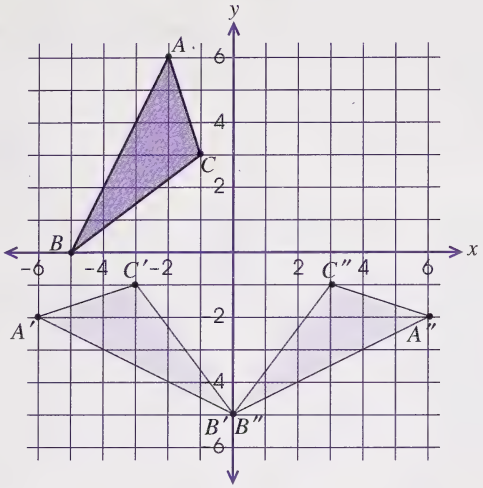
5. Would the final figure (figure C) be different if you applied the mapping rules from questions 3.a. and 3.b. in reverse order? Explain.

6. Describe the single transformation, first in words and then as a mapping rule, that would map the original figure onto the final image for each of the following.

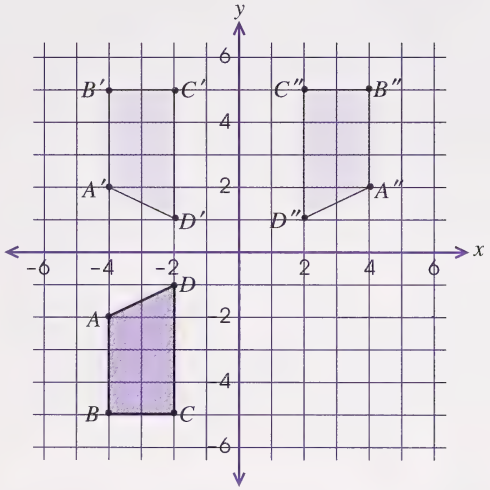
a.



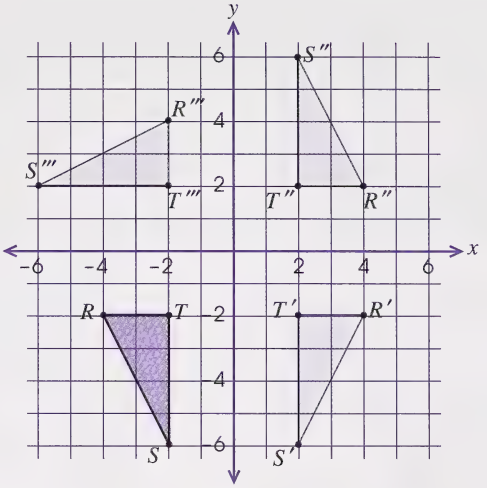
b.



c.



d.



7. What transformation is indicated by each of the following mapping rules? State the rotations in terms of clockwise turns.

a. $(x, y) \rightarrow (-x, -y)$

b. $(x, y) \rightarrow (x-2, y+3)$

c. $(x, y) \rightarrow (-x, y)$

d. $(x, y) \rightarrow (-y, x)$

e. $(x, y) \rightarrow (x, -y)$

f. $(x, y) \rightarrow (y, -x)$

8. Copy and complete the following table by indicating what kind of transformation would map the coordinates of the original triangle onto the coordinates of the image.

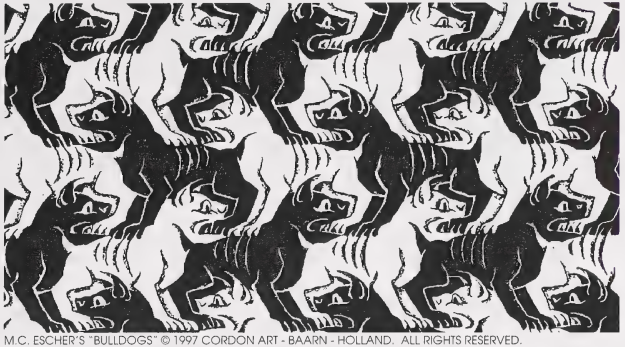
Coordinates of the Original			Coordinates of the Image			Type of Transformation
A	*B*	*C*	*A′*	*B′*	*C′*	
(2, 5)	(3, 4)	(6, 2)	(3, 3)	(4, 2)	(7, 0)	
(−3, 2)	(−5, 6)	(−1, 4)	(3, −2)	(5, −6)	(1, −4)	
(1, 0)	(−2, 5)	(−4, −3)	(−1, 0)	(2, 5)	(4, −3)	
(4, 3)	(5, −2)	(2, 1)	(−3, 4)	(2, 5)	(−1, 2)	
(−3, −5)	(−2, 1)	(−4, 3)	(−3, 5)	(−2, −1)	(−4, −3)	

Check your answers by turning to the Appendix.

Now Try This

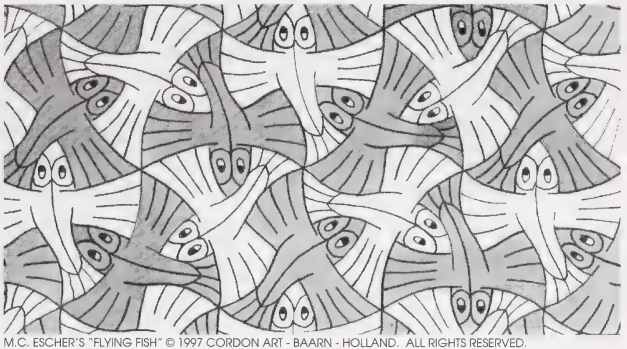
9. Maurits Escher is famous for combining art and mathematics in his designs. Study the following Escher designs. What combination of transformations are used to create each design?

a.

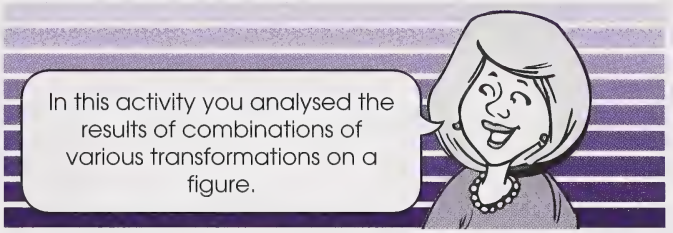


M.C. ESCHER'S "BULLDOGS" © 1997 CORDON ART - BAARN - HOLLAND. ALL RIGHTS RESERVED.

b.



M.C. ESCHER'S "FLYING FISH" © 1997 CORDON ART - BAARN - HOLLAND. ALL RIGHTS RESERVED.

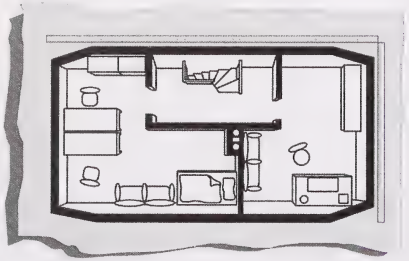


Activity 5: Enlargements and Reductions (Dilatations)

Have you ever had a photograph enlarged? In the photographs given, notice that only the size of each figure in the photograph was affected when it was enlarged—the shape remained unchanged. In fact, when anything is enlarged or reduced, the size changes and the shape remains the same.

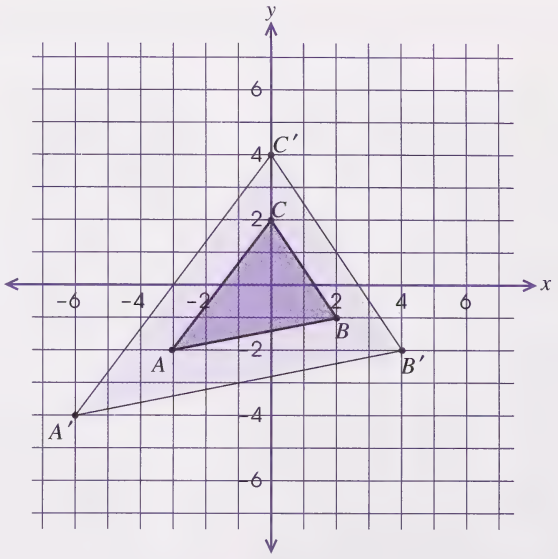
In mathematics, when a figure is enlarged or reduced, the transformation is called a **dilatation**.

Some people's lives revolve around dilatations. For instance, an architect—who creates or works with blueprints, floorplans, and models of houses and buildings—makes a living working with dilatations.



Complete the following questions to help you explore dilatations.

1. ΔABC—with coordinates $A(-3,-2)$, $B(2,-1)$, and $C(0,2)$—and $\Delta A'B'C'$—with coordinates $A'(-6,-4)$, $B'(4,-2)$, and $C'(0,4)$—are given on the following coordinate grid.



a. Measure and compare the lengths of the corresponding sides of ΔABC and $\Delta A'B'C'$.

b. Measure and compare the corresponding angles of ΔABC and $\Delta A'B'C'$.

c. What else can you say about the corresponding sides of the two triangles?

Check your answers by turning to the Appendix.

Properties of a Dilatation

- The image is similar to the original figure.
- The orientation of the image and the original figure are the same.
- The corresponding sides of the image and the original figure are parallel.

The **scale factor** of a dilatation is the ratio of the lengths of the sides of the image to the lengths of the sides of the original figure. You can use a mapping rule to give a scale factor.

$(x, y) \rightarrow (kx, ky)$, where k is any number

2. What is the scale factor for the diagram in question 1?

3. Write the mapping rule for the dilatation in question 1.

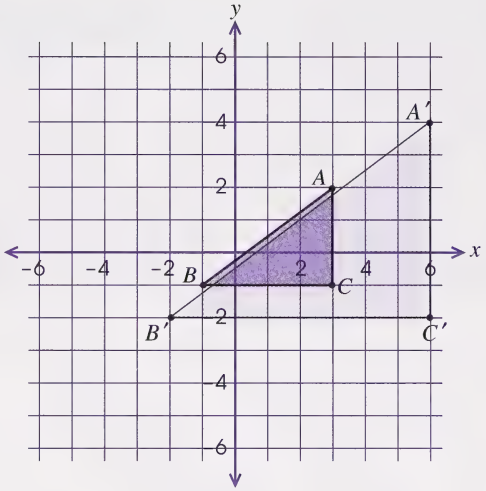
4. Figure *WXYZ* has vertices $W(-1, 2)$, $X(3, 2)$, $Y(3, -1)$, and $Z(-2, -1)$. Draw the figure on a coordinate grid. Using a scale factor of 2, write the coordinates of and draw the image of *WXYZ*.

5. **a.** Draw ΔDEF with vertices $D(0, 4)$, $E(-2, 0)$, and $F(2, 0)$ on a coordinate grid; then find its image using a scale factor of $\frac{1}{2}$.

b. Write the mapping rule for the dilatation in question 5.a.

6. Figure *ABCD* has vertices $A(-1, 1)$, $B(-1, -2)$, $C(2, -2)$, and $D(2, 1)$. Draw *ABCD* on a coordinate grid and find its image under the mapping rule $(x, y) \rightarrow (3x, 3y)$.

7. A dilatation $(x, y) \rightarrow (2x, 2y)$ is applied to ΔABC to get $\Delta A'B'C'$.



a. Find the areas of ΔABC and $\Delta A'B'C'$.

b. Write a statement about the relationship between the two areas in question 7.a.

8. a. When a dilatation is applied to a figure, what changes?

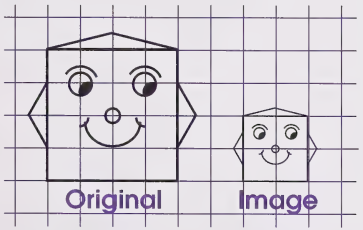
b. When a dilatation is applied to a figure, what remains the same?

9. Copy and complete the following chart in your notebook. Fill in the blanks for dilatations of points about $(0, 0)$.

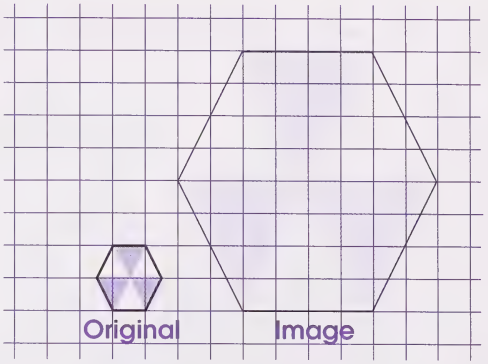
Original	Image	Mapping Rule
$(3, -1)$		$(x, y) \rightarrow (2x, 2y)$
$(4, 2)$	$(2, 1)$	
	$(-2, 3)$	$(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$
$(4, 6)$	$(6, \quad)$	
$(\quad, 2)$	$(9, \quad)$	$(x, y) \rightarrow (3x, 3y)$

10. A figure and its image are shown in each of the following pairs. Determine the scale factor.

a.



b.



11. Copy and complete the following chart in your notebook.

Area of Original	Area of Image	Scale Factor
$24\ cm^2$	$6\ cm^2$	
$16\ cm^2$	$36\ cm^2$	
	$4\ cm^2$	$\frac{1}{3}$
$8\ cm^2$		2.5

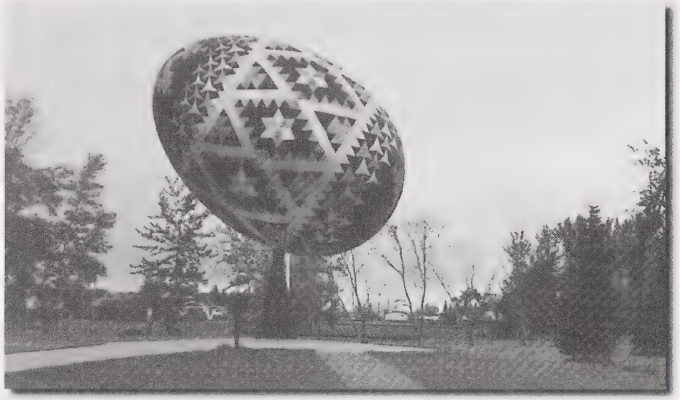
Check your answers by turning to the Appendix.

Now Try This

Use one of the problem-solving strategies to solve question 12.

You have discovered that dilatations produce images that are similar in shape to the original but differ in size.

12. The Easter egg in Vegreville, Alberta, is an example of a real-world dilatation. The monument is an enlargement of an actual Easter egg. Give four real-world examples of dilatations. State whether each is an enlargement or a reduction.



Check your answer by turning to the Appendix.

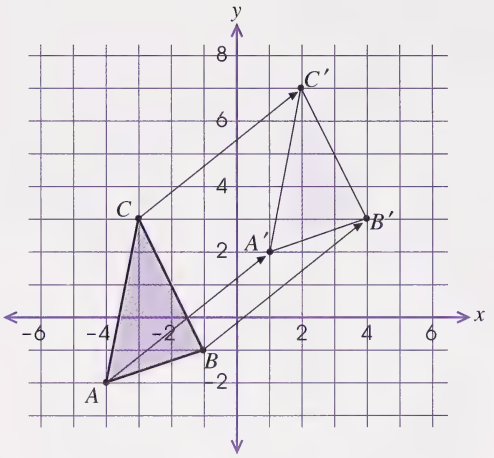
Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

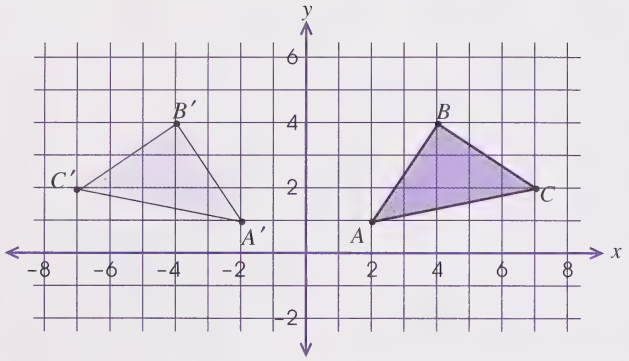
In this section you analysed four common types of transformations. These four transformations are illustrated in the following diagrams along with the mapping rule for each.

Translation



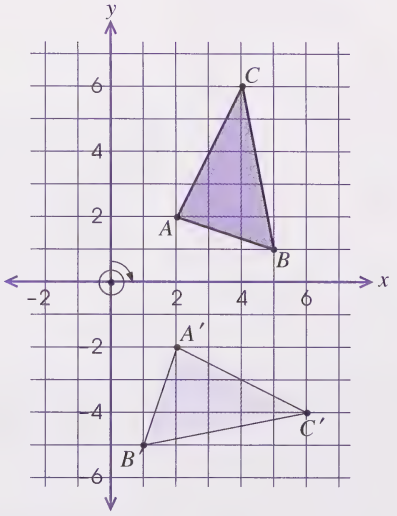
ΔABC is translated 5 units to the right and 4 units up. The mapping of the vertices is $A(-4,-2) \rightarrow A'(1,2)$, $B(-1,-1) \rightarrow B'(4,3)$, and $C(-3,3) \rightarrow C'(2,7)$. Thus, the mapping rule is $(x, y) \rightarrow (x+5, y+4)$.

Reflection



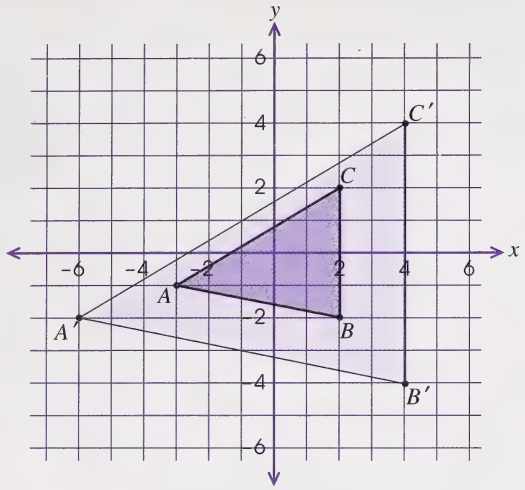
ΔABC is reflected about the y-axis. The mapping of the vertices is $A(2,1) \rightarrow A(-2,1)$, $B(4,4) \rightarrow B'(-4,4)$, and $C(7,2) \rightarrow C'(-7,2)$. The mapping rule is $(x, y) \rightarrow (-x, y)$.

Rotation



ΔABC is rotated 90° clockwise about the origin. The mapping of the vertices is $A(2,2) \rightarrow A'(2,-2)$, $B(5,1) \rightarrow B'(1,-5)$, and $C(4,6) \rightarrow C'(6,-4)$. Thus, the mapping rule is $(x, y) \rightarrow (y, -x)$.

Dilatation



ΔABC is enlarged to become $\Delta A'B'C'$. The mapping of the vertices is $A(-3,-1) \rightarrow A'(-6,-2)$, $B(2,-2) \rightarrow B'(4,-4)$, and $C(2,2) \rightarrow C'(4,4)$. Thus, the mapping rule is $(x, y) \rightarrow (2x, 2y)$.

1. A quadrilateral has vertices $A(-1,-2)$, $B(-1,2)$, $C(2,1)$, and $D(2,-1)$.

a. Draw the quadrilateral on a coordinate grid.

b. Translate the original quadrilateral 4 units to the left and 5 units up.

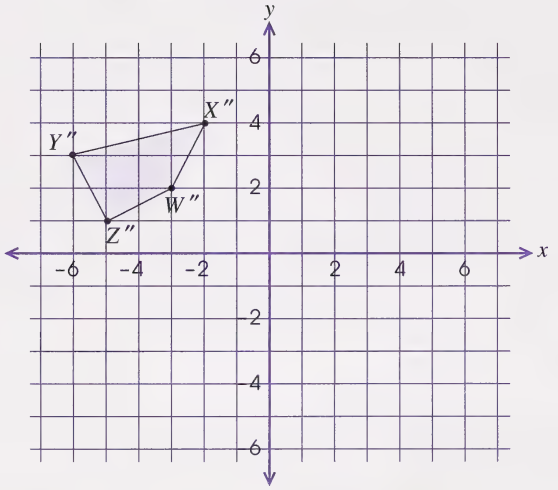
c. Rotate the original quadrilateral 270° ccw about $(0,0)$. What other rotation is this equivalent to?

d. Write the mapping rules for questions 1.b. and 1.c.

e. Reflect the original quadrilateral about the line $y = -3$.

2. ΔABC has coordinates $A(-5,2)$, $B(-2,3)$, and $C(-2,6)$. Describe a transformation that would put ΔABC entirely in the third quadrant.

3. Quadrilateral $W''X''Y''Z''$ is the result of a translation of 1 unit to the right and 5 units up and a reflection about the y-axis. Draw the quadrilateral in its original position.



4. Quadrilateral *MNOP* has coordinates $M(-1,-2)$, $N(-1,2)$, $O(2,0)$, and $P(2,-2)$. Apply the following mapping rules to quadrilateral *MNOP*, and write the new coordinates. Draw both the original quadrilateral and the new quadrilateral for each mapping.

   a. $(x,y)\rightarrow(2x,2y)$ **b.** $(x,y)\rightarrow(x-2,y+3)$

   c. $(x,y)\rightarrow(-x,y)$

5. Some letters in the alphabet are reflections about a vertical or horizontal line, dividing the letters into two equal parts. For instance, look at the letter M.

   List other letters of the alphabet that are reflections in a vertical (such as the letter M) or horizontal line. Draw the lines of reflection in each letter; then group the letters as having only a vertical line of reflection, only a horizontal line of reflection, or both.

Check your answers by turning to the Appendix.

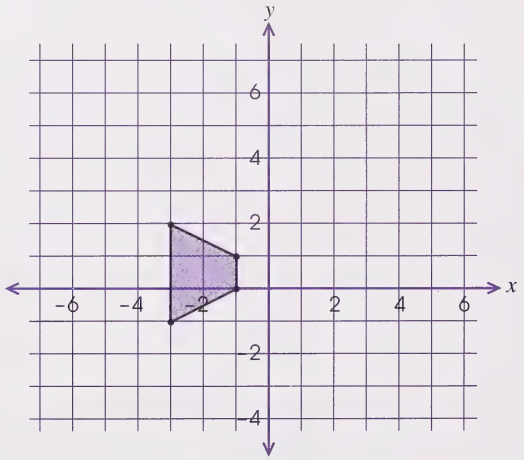
If you have access to a computer, you may wish to view the software program *Understanding Geometry* (Neufeld and Associates) for additional help with transformations. Do Part 2: Transformations, and answer as many of the Practice questions as you feel are necessary. This program may be available at your school or local library.

Enrichment

You have looked at some translations, reflections, rotations, and dilatations on a standard grid. These are not the only kinds of transformations.

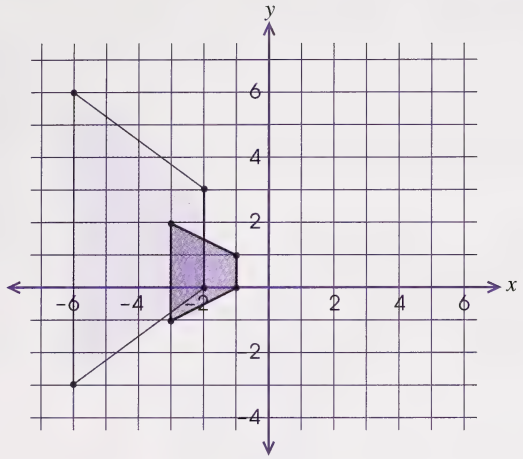
Example 1

What happens to the following figure when you apply the mapping rule $(x,y)\rightarrow(2x,3y)$?

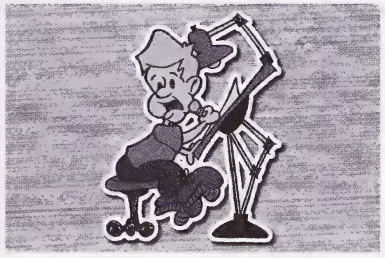


Solution

Plot the figure on a coordinate grid, and apply the mapping rule to find its image.

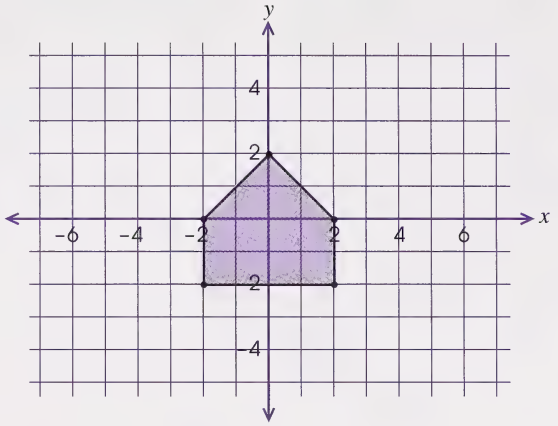


The figure has been enlarged and stretched to produce the image.



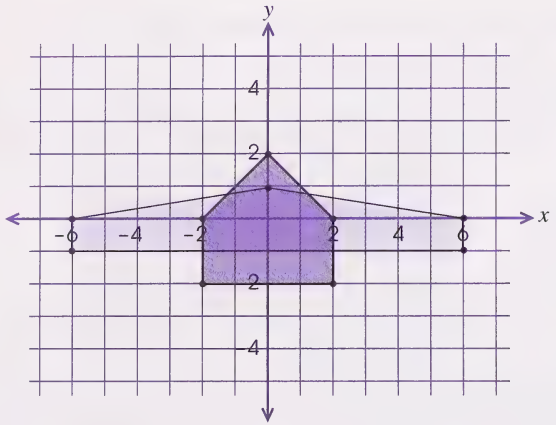
Example 2

Find the image of the following figure under the mapping rule $(x, y) \rightarrow \left(3x, \frac{y}{2}\right)$.



Solution

Plot the figure on a coordinate grid, and apply the mapping rule to find its image.



The figure has been reduced vertically and stretched horizontally.

Examples 1 and 2 are examples of distortions. You may stretch, shrink, turn, and do many other things to a figure by applying a variety of mapping rules. This is similar to what happens to your image as you stand in front of the various mirrors in a fun house. Distortions of one kind or another are often used by cartoonists, artists, and designers to produce special effects.

1. a. Draw rectangle *ABCD* with coordinates $A(0,0)$, $B(5,0)$, $C(5,3)$, and $D(0,3)$ on a coordinate grid; then draw its image using the mapping rule $(x,y)\rightarrow\left(x,\frac{x+y}{2}\right)$.

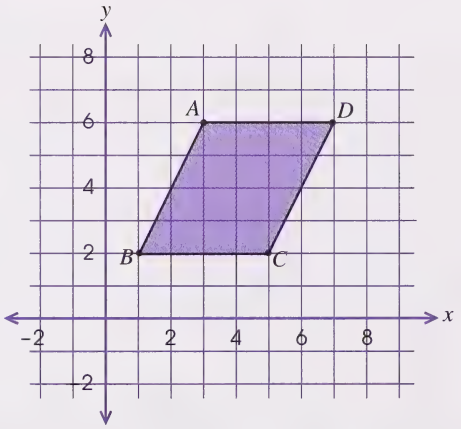
b. What effect does the mapping rule in question 1.a. have on rectangle *ABCD*?

2. a. Draw square *ABCD* with coordinates $A(5,3)$, $B(1,3)$, $C(1,-1)$, and $D(5,-1)$ on a coordinate grid; then draw its image using the mapping rule $(x,y)\rightarrow(-x,2y)$.

b. What effect does the mapping rule have on square *ABCD*?

c. Explain what would happen to square *ABCD* when drawn using the mapping rule $(x,y)\rightarrow(2x,-y)$.

3. Explain what would happen to figure *ABCD* under each of the following mappings.



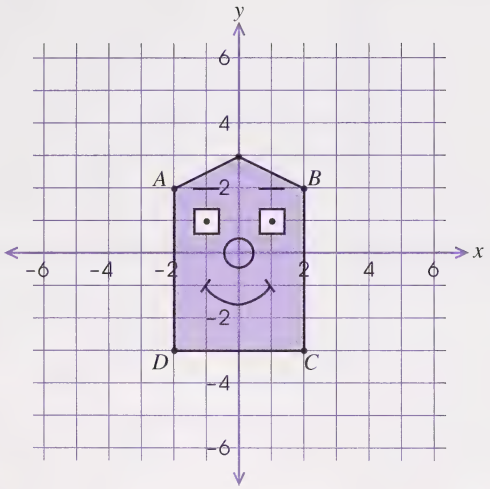
a. $(x,y)\rightarrow\left(2x,\frac{x+y}{2}\right)$

b. $(x,y)\rightarrow\left(-x,\frac{1}{2}y\right)$

c. $(x, y) \rightarrow (3x, y)$

d. $(x, y) \rightarrow (-2x, -2y)$

4. On a coordinate grid, find the image of the following figure using the mapping rule $(x, y) \rightarrow (2x, x^2 + y)$. Use the rule for points *A*, *B*, *C*, and *D*, the pupils of the eyes, and the corners of the mouth. Extend the eyebrows, squares around the eyes, and the nose as you think these parts would distort.



Check your answers by turning to the Appendix.

You can also distort a figure by changing the scale on the grid or by distorting the grid.

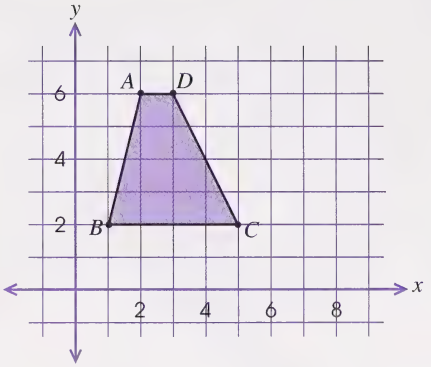
Example 3

Using vertices $A(2, 6)$, $B(1, 2)$, $C(5, 2)$, and $D(3, 6)$, plot figure *ABCD* on coordinate grids with various scales for x and y. What happens to figure *ABCD*?

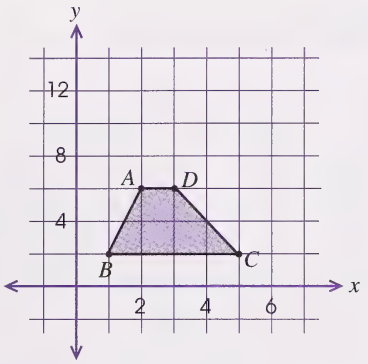
Solution

Plot the vertices of *ABCD* on each coordinate grid and notice the effects.

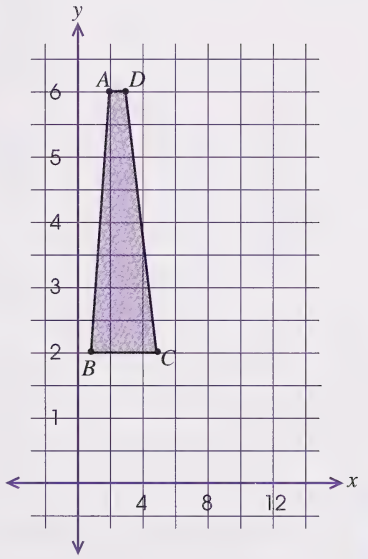
The following is a standard scale on the grid.



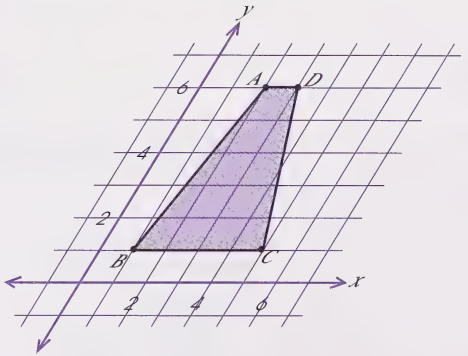
The coordinate grid on the right has 1 unit on the x-axis equalling 2 units on the y-axis. Notice that the figure is flattened.



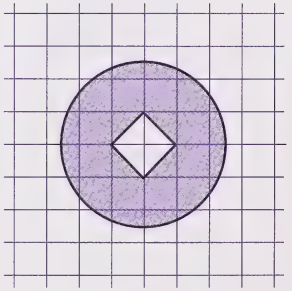
The coordinate grid on the right has 4 units on the x-axis equalling 1 unit on the y-axis. Notice that the figure is stretched vertically and squeezed horizontally.



The following coordinate grid has the vertical axis slopes to the right. Notice that the figure is slanted to the right.



5. Find the page with three distorted grids for this question at the back of the Appendix. Either photocopy the grids or pull out the page. Then draw the following figure on each of the grids.



Check your answer by turning to the Appendix.

If you have access to a graphing program, you may wish to use it to explore transformations. The following are general instructions for the *Zap-a-Graph*™ program using a Macintosh computer.

Step 1: Double-click on the *Zap-a-Graph*™ icon to open the application.

Step 2: Go to the Define menu; then select Point.

Step 3: Type in the coordinates of a point you wish to use; pick the largest dot size; and plot the point. There should be a point showing on your graph.

Step 4: Go to the Transform menu; then select Reflect in x axis. There should be a point showing on the opposite side of the x-axis from the original point now.

Step 5: Go to the Erase menu; then select Current graph. Only the original point should be showing on the graph now.

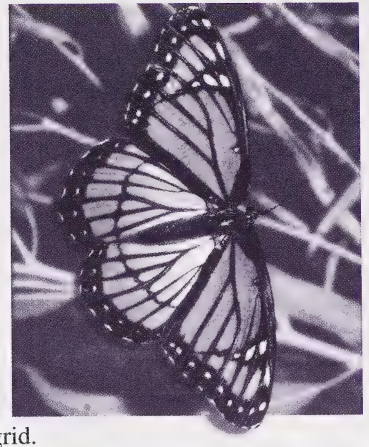
Step 6: Repeat Steps 4 and 5 with various other transformations. Note what happens to the point after each transformation.

Step 7: Repeat Steps 1 to 5 using a line graph. Note what happens to the graph after each transformation.

6. Are the results of the transformations created by the graphing program similar to those you examined in this section?

Check your answers by turning to the Appendix.

Conclusion



In this section you expanded your knowledge of transformations by working with lines and shapes (mostly triangles) on a coordinate grid.

You translated, reflected, and rotated figures (singly and in combination) and discovered some of the properties and characteristics of these transformations. You also discovered what dilatations are and how to complete these transformations on a coordinate grid.

After completing this section you may never view people, animals, objects, or even scenery in nature the same way again. For instance, look at the butterfly in the photograph. Although you may admire its beauty first, you will also probably notice how it is an excellent example of transformations (particularly reflections).

Assignment

Assignment Booklet

You are now ready to complete the assignment for Section 1.

Section 2: Measurement and 2-D and 3-D Figures

Would you like a career in computer-assisted design? Computers are used to design everything from nuts and bolts, to vehicles and vehicle parts, to floor plans and houses. A computer-assisted design (CAD) program can be used to draw an object in three dimensions and then used to rotate the object so all sides and angles can be viewed.

Can you visualize objects in two and three dimensions? The ability to visualize objects in two dimensions, and particularly in three dimensions, is important in designing vehicles, buildings, and even landscapes.

In this section you will analyse figures in two and three dimensions. First, you will analyse what happens when a locus of points is created by moving a point or figure in a plane. Next, you will analyse different views of three-dimensional block diagrams by drawing several two-dimensional views of the diagram, by drawing a three-dimensional view of the diagram, and by creating a three-dimensional model. You will also apply measurements to three-dimensional figures (such as pyramids and cones) and calculate their volume. Finally, you will analyse the relationship between surface area and volume, and you will use your understanding of this relationship to analyse design in the packaging industry.

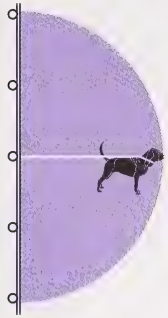
Activity 1: Spatial Problems in Two Dimensions

Locus of Points

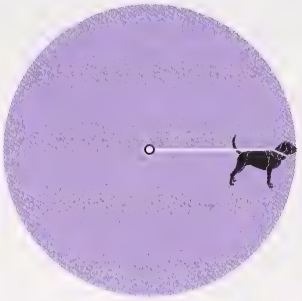
Have you ever seen a dog tied to a fence or lamp post? Did you know that the area over which the dog can walk while tied up can be defined as a **locus of points**?

A locus of points is the set of all points in a plane that satisfy a given condition.

In the case of the dog on a leash, the locus is the set of all points of the area that the dog can reach while tied up. This locus may look something like the following.



The locus is the set of all points in a plane in the semicircular area on one side of the fence.



The locus is the set of all points in a plane around a post.

Many practical problems involve the idea of a locus.

Example 1

A gorilla is to be housed in a rectangular enclosure that is 3.6 m by 4.5 m. For the safety of zoo patrons, a second rectangular fence is to be built around the gorilla's enclosure. Draw a diagram to show the minimum area between the inner fence and the outer fence if the gorilla can reach 0.8 m outside its enclosure. Then calculate the area between the outer fence and the inner fence. Round your answer to the nearest tenth of a metre.

Solution

Area = Area of Outer Fence – Area of Inner Fence

$$= [(3.6 + 0.8 + 0.8)(4.5 + 0.8 + 0.8)] - (3.6)(4.5)$$
$$= (5.2)(6.1) - (3.6)(4.5)$$
$$= 31.72 - 16.2$$
$$= 15.52$$

The area between the outer fence and the inner fence is approximately $15.5\ \text{m}^2$.

Example 2

A sheep is tied to a fence with a rope that is 15 m long. Sketch the locus that shows the area of grass the sheep can eat; then calculate the area of grass that can be eaten. Round your answer to the nearest square metre $\left(\text{m}^2\right)$.

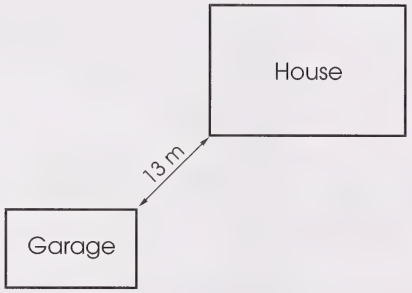
Solution

$$\text{Area} = \frac{1}{2}\pi r^2$$
$$= \frac{1}{2}\pi(15)^2$$
$$= \frac{1}{2}\pi(225)$$
$$\doteq 353.4$$

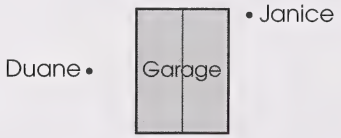
The area the sheep can graze is approximately $353\ \text{m}^2$.

1. An FM radio transmitter sends out its signal a distance of 85 km. Choose a suitable scale, and draw the locus that represents the area covered. Describe the area covered by the transmitter assuming there are no high hills or mountains within the area to block the transmission of the FM signals.

2. Andrea wants to plant a birch tree at least 8 m from her house and her garage, which are located as shown. Sketch the area where the tree cannot be planted.



3. Janice is trying to find her friend Duane, who is hiding behind a garage (as shown in the overhead view).



a. Sketch the locus of all the points from which Janice would not be able to see Duane.

b. Measure the angle of Duane's obstructed view.

Check your answers by turning to the Appendix.

A locus of points does not have to be an area. It can be all the points along a line (such as the centre line on a highway), or it can be individual points (such as those created by a spot on a car tire as the tire rolls down a highway).

4. Sketch and describe each locus in a plane.

a. all pairs of points of equal distance from the centre of the back of a hockey net to the blue line

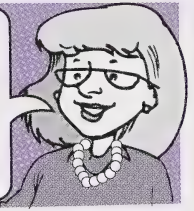
b. all points created by a paint spot on a car tire as it moves along the highway

5. Two sleeping cabins, *M* and *N*, at a campground are located 30 m apart and in a straight line. A shower building is to be located the same distance from each cabin and at least 10 m north of both cabins. Sketch a diagram showing the locus where the shower building could be located.

Check your answers by turning to the Appendix.

Relating Perimeter and Area to Length

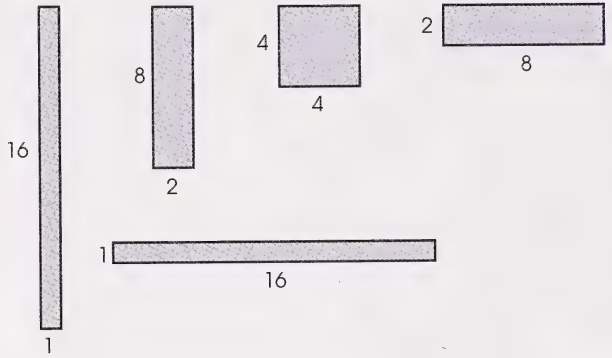
In this part of the activity you will discover how the perimeter and area of a rectangle are related to the length of the rectangle. You will also use graphs to discover more about how the perimeter and area of a rectangle are related.



Consider the following question.

How many rectangles can you form that have an area of 16 square units?

6. There are many rectangles with an area of 16 square units. Here are some of the possibilities.



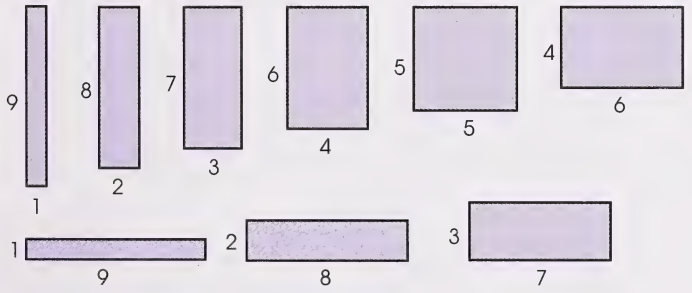
a. Use the given rectangles to complete the following table.

Rectangles with an Area of 16 Square Units		
Length (units)	Width (units)	Perimeter (units)
1	16	
2	8	
4	4	
8	2	
16	1	

b. Which rectangle or rectangles have the greatest perimeter?

c. Which rectangle or rectangles have the least perimeter?

7. There are many rectangles with a perimeter of 20 units. Here are some of the possiblilities.



a. Use the given rectangles to complete the following table.

Rectangles with a Perimeter of 20 Units		
Length (units)	Width (units)	Area (square units)
1	9	
2	8	
3	7	
4	6	
5	5	
6	4	
7	3	
8	2	
9	1	

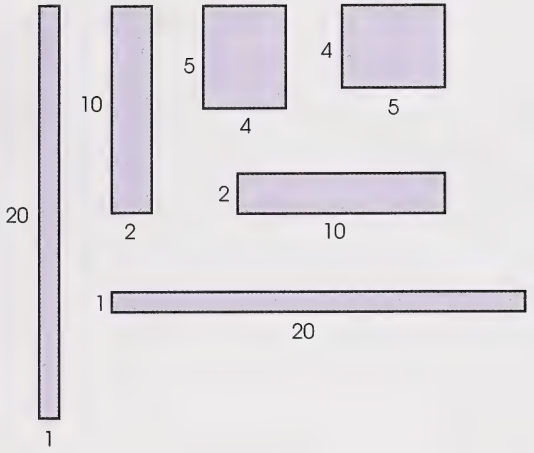
b. Graph the set of ordered pairs, (length, area), for the rectangles with a perimeter of 20 units.

c. Because a rectangle with a perimeter of 20 units can have fractional units for its length, connect the dots on the graph in question 7.b.

d. What does the graph tell you about the relationship between length and area?

e. What do you notice about the rectangle with the greatest area?

8. There are many rectangles with an area of 20 square units. Here are some possibilities.



a. Use the given rectangles to complete the following table.

Rectangles with an Area of 20 Square Units

Length (units)	Width (units)	Perimeter (units)
1	20	
2	10	
4	5	
5	4	
10	2	
20	1	

b. Graph the set of ordered pairs, (length, perimeter), for the rectangles with an area of 20 square units.

c. Because a rectangle with an area of 20 square units can have fractional units for its length, connect the dots on your graph in question 8.b.

d. What does the graph tell you about the relationship between length and perimeter?

9. Optek has decided to build a rectangular dog run. He wants to use as little fencing as possible. If his dog needs about $144\ m^2$, how long and wide should the dog run be?

10. Arlene wants to build a rectangular pen for her chickens. The chickens need an area of 36 m^2. To save on fencing, she plans to use a river that runs through her farm as one side of the pen.

a. Draw and label all the possible designs for the pen.

b. Which is the most economical design for the pen?

c. Which is the most expensive design for the pens?

11. A hardware store sells fencing by the metre. If Mr. Petruk purchases 42 m of fencing to build a rectangular pen, then what will the dimensions of the pen be if the length is 1 m more than the width?

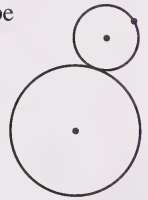
12. Suppose you have 16 m of fencing. Design the largest pen possible with this amount of fencing. Your design can be any shape.

Check your answers by turning to the Appendix.

Now Try This

Use one of the problem-solving strategies to answer the following problem.

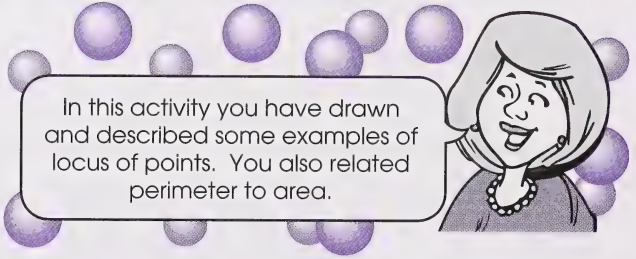
13. Use the diagram on the right to sketch and describe the following:



a. all points traced by the centre of the smaller circle as it rotates around the larger circle

b. all points traced by a point on a small circle as it rotates around the larger circle

Check your answers by turning to the Appendix.



Activity 2: Spatial Problems In Three Dimensions

Do you remember constructing with building blocks when you were younger? If so, did you pay any attention to the different views (front, right side, left side, back, or top)? Were these views different?

When designing house plans, an architect may have to draw the views of the front, the back, and both sides. People involved in the design of cars, machine parts, or even nuts and bolts need to be able to draw various views of their product.

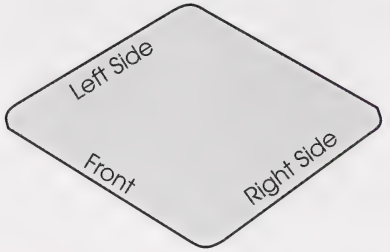
In this activity you will discover how to draw different views of 3-D models in two dimensions as well as 3-D drawings from 2-D plans.

Drawing 2-D Plans from Models

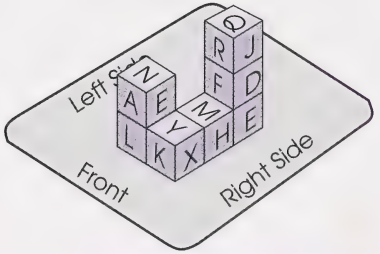
In this exercise you will build a model from blocks and draw the various views using square dot paper and isometric dot paper.

Step 1: Obtain a set of blocks used in mathematics or some building blocks. (You can even use sugar cubes.) It is important that each block is a cube.

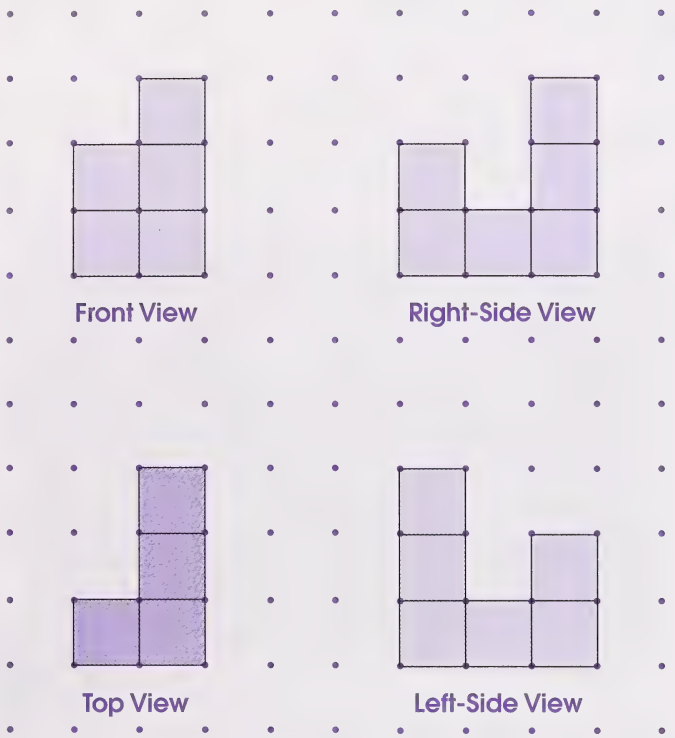
Step 2: Obtain a piece of cardboard to use as a mat. Label the front, the left side, and the right side of the mat.



Step 3: Use the blocks to build a model on the mat so it looks as follows:



Step 4: Copy or tear out a page of the square dot paper from the Appendix and draw and label, in two dimensions, the front, top, right-side, and left-side views of the model. Your drawings should look as follows:



1. Are the right-side and left-side views the same?

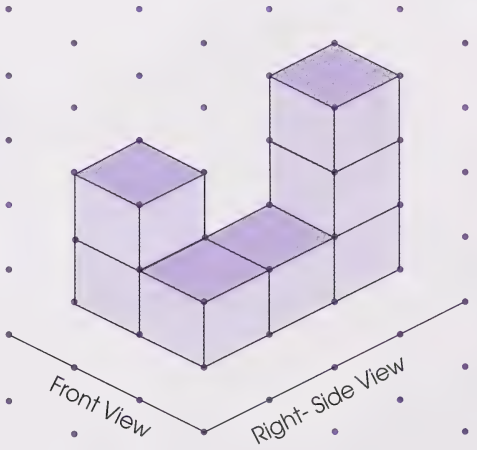
2. How can you make the left side fit the right side?

3. How are the figures labelled Right-Side View and Left-Side View related?

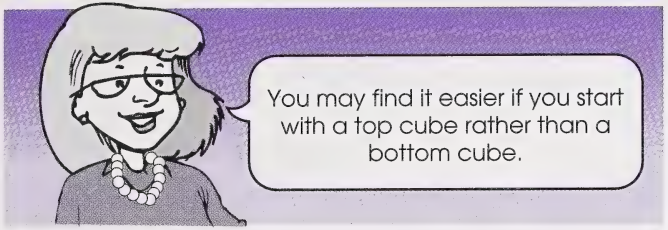
Check your answers by turning to the Appendix.

Copy or tear out a page of the isometric dot paper from the back of the Appendix. Draw a three-dimensional (3-D) drawing of the model. Label the front and right-side views.

Your drawing should be set up as follows:



4. Was it difficult to draw the cubes without putting in the hidden lines?

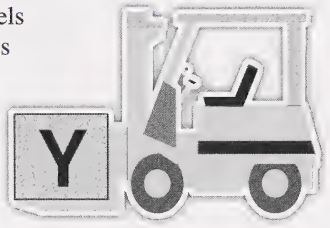


5. Build other models using your blocks (or sugar cubes). Draw the top, right-side, and left-side views on square dot paper and the complete model on isometric dot paper. Do you find it easier to draw the different views the more you practise?

Check your answers by turning to the Appendix.

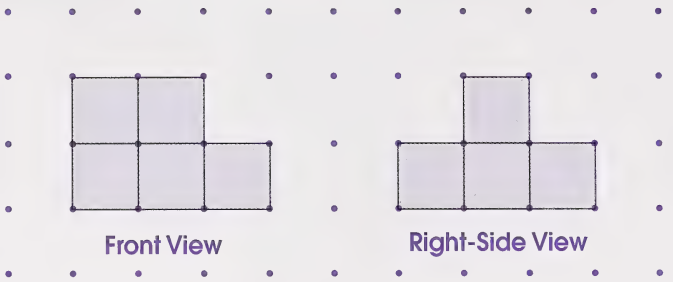
Building Models and Drawing 3-D Plans from 2-D Plans

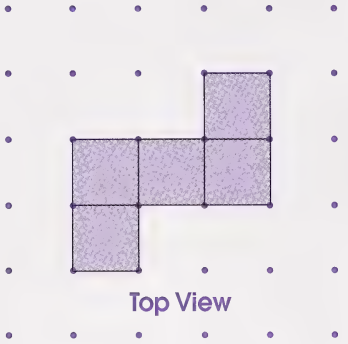
Now you will try to build block models and make three-dimensional drawings from two-dimensional plans.



Example 1

Use the following two-dimensional views to construct a three-dimensional model with your building blocks or sugar cubes.



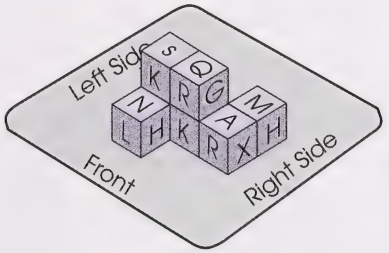


Solution

First, look at all three views. This will help you to decide which rows have the most blocks.

Step 1: The top view is the same as the base since you cannot have a block with nothing below it. Start by laying out all the blocks in the top view as the base.

Step 2: Use the front view and the right-side view to determine where and how many blocks to add on top of the base. The front view shows there are two blocks on the left side, while the right-side view shows these two blocks must lie down the centre of the base. This completes the construction.

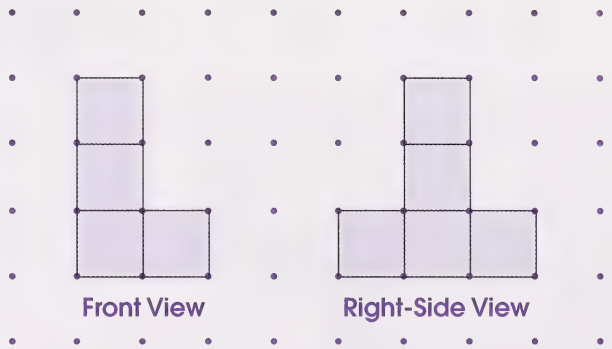


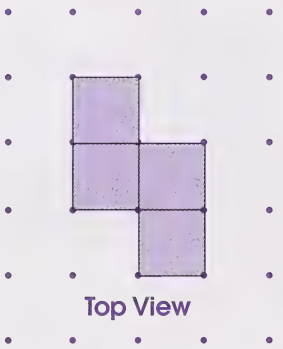
6. Why was the left-side view not included in Example 1?

Check your answer by turning to the Appendix.

Example 2

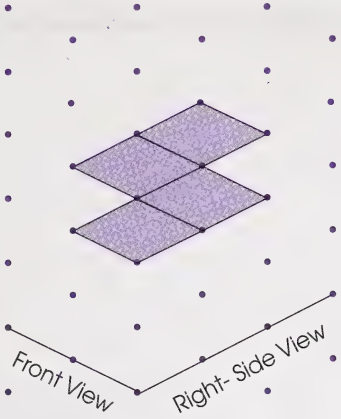
Draw the 3-D model, on isometric dot paper, based on the following set of views.



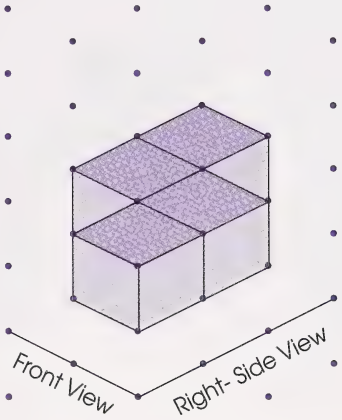


Solution

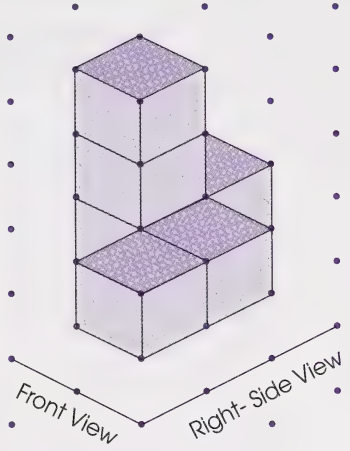
Step 1: Draw a two-dimensional drawing of the top view.



Step 2: Complete the four cubes in the base of the isometric view. Your completed cubes of the top view should look as follows:



Step 3: Complete the isometric drawing by adding in the remaining cubes required for the front and right-side views. Your completed model should look as follows:

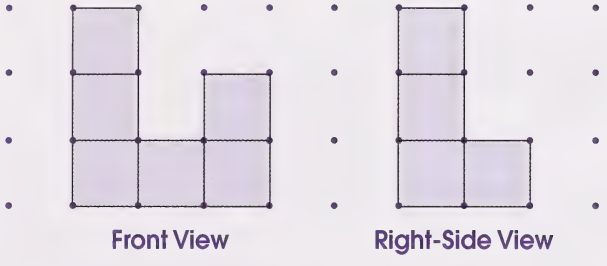


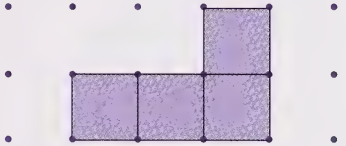
You will find it necessary to erase some of the hidden lines as you add cubes.

7. Draw the model of Example 1 on isometric dot paper using the three 2-D views.

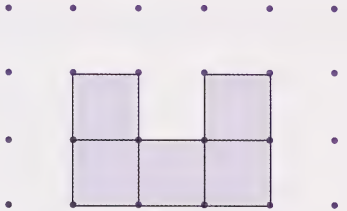
8. Make 3-D drawings, on isometric dot paper, of the following sets of 2-D views.

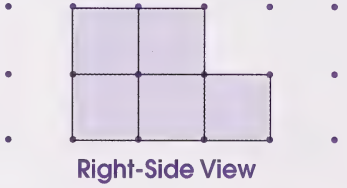
a.

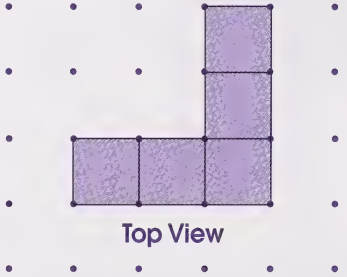




b.







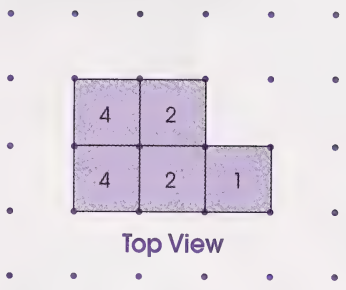
9. Construct models from the plans in questions 8.a. and 8.b. using blocks or sugar cubes.

Check your answers by turning to the Appendix.

Sketches can also be made with the top view as a base plan with numbers on each block. The numbers tell you how many blocks are in that position.

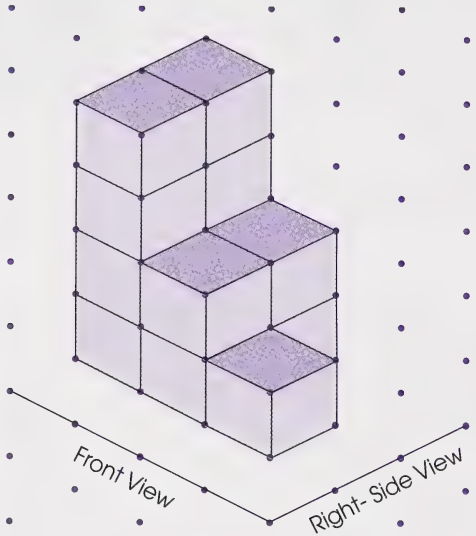
Example 3

Use the following numbered-base plan to draw a model on isometric dot paper; then construct a block or sugar-cube model. **Note:** The bottom of the numbered-base plan is the front.



Solution

The model of the base plan, drawn on isometric dot paper, is as follows:



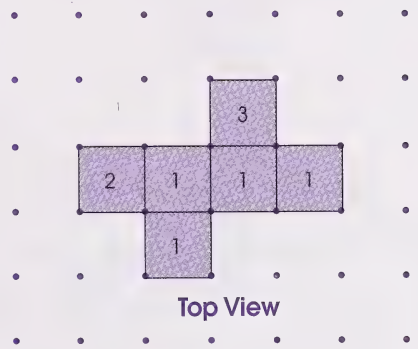
The constructed model, using blocks or sugar cubes, will look similar to the drawing on isometric dot paper.

The two left columns each have four blocks in them, since there is a 4 on each of the left-most blocks in the base plan. The middle columns each have two blocks, since there is a 2 on each of the centre blocks in the base plan. There is only one block on the right side, since there is a 1 on that block in the base plan.

10. Using isometric dot paper, draw models of the following base plans.

a.

b.

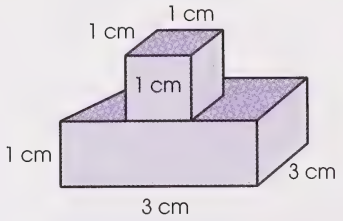


c. Rotate the base plan in question 10.a. 90° ccw so that the three blocks are in front. Draw the model on isometric paper and compare it to the drawing you did for question 10.a.

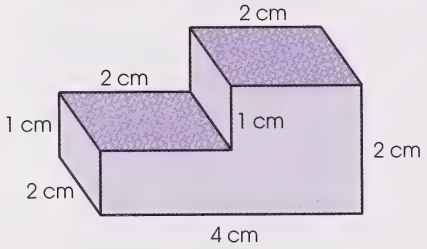
d. Construct a model from blocks or sugar cubes for question 10.a. Is the same model correct for the rotated drawing in question 10.c.?

11. Draw and label the front view, right-side view, left-side view, and top view of the following.

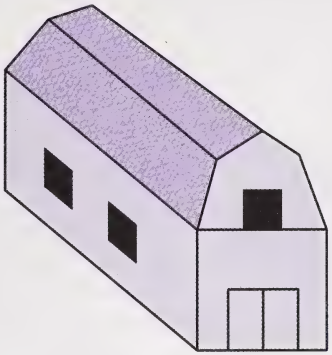
a.



b.



12. Sketch the front view, the right- and left-side views, and the top view of the following. **Note:** Assume the right-side view is the same as the left-side view.



Check your answers by turning to the Appendix.

Now Try This

Draw your own two-dimensional views on square dot paper; then make drawings on isometric dot paper and construct models using blocks or sugar cubes.

If you have access to a computer-assisted design (CAD) program, you may wish to try drawing some of the sketches and views from questions 11 and 12.

In this activity you have drawn 2-D plans from 3-D models and 3-D models from 2-D plans. By doing this, you extended your ability to visualize in three dimensions.

Activity 3: Volumes of Pyramids and Cones

What is the volume of air in one of the pyramids at the Muttart Conservatory in Edmonton? What is the volume of your favourite snow cone? To answer these questions you need a formula to calculate the volumes.

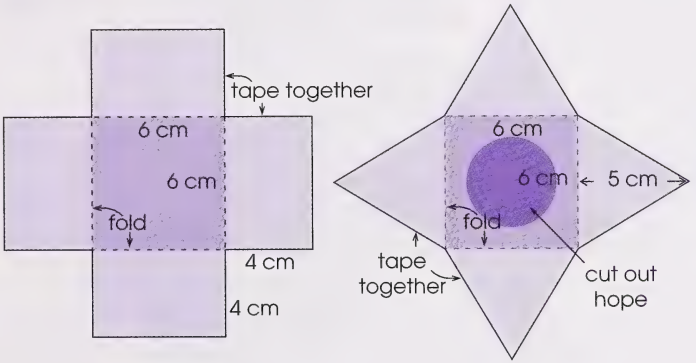
Volume of a Pyramid

You know that volume is measured in cubic units and that the volume of a rectangular prism is found using the formula $V = BH$, where B is the area of the base and H is the height.

> **Remember:** The area of any right prism equals the area of the base × height.

See if you can discover a formula for finding the volume of a pyramid as you try the following activity.

Step 1: Cut out the nets given at the back of the Appendix, or make your own nets for a rectangular prism and a rectangular pyramid like the ones shown. Glue the page with the nets to construction paper or boxboard (shoe or cereal boxes).



Step 2: Fold and tape the sides of the net together (as indicated) to make an open-topped rectangular prism.

Step 3: Cut out the circle in the middle of the net for the rectangular pyramid. Then fold and tape the edges together (as indicated).

1. Are their heights the same?

Check your answer by turning to the Appendix.

Step 4: Fill the pyramid with rice (or any other small-grained, pourable material); then pour it into the rectangular prism.

2. How many times must you repeat Step 4 in order to completely fill the rectangular prism?

3. What does this tell you about the relationship between the volume of a prism and the volume of a pyramid that have the same base and the same height? Write a formula for the volume of a rectangular pyramid based on your findings.

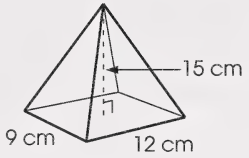
Check your answers by turning to the Appendix.

You found that the volume of a rectangular pyramid is one-third the volume of a rectangular prism with the same base and height. You could conduct other experiments with prisms and pyramids with bases shaped like triangles (or any regular polygon) and find that the results are the same.

The volume of any right pyramid is found by using the general formula $V = \frac{1}{3}BH$ ($\frac{1}{3} \times$ area of the base $\times$ the height of the pyramid).

Example 1

Find the volume of the pyramid given.



Solution

Since $V = \frac{1}{3}BH$, you must determine B first.

$$\begin{aligned} B &= \ell w \\ &= 12(9) \\ &= 108 \end{aligned}$$

In this case, the base is a rectangle; thus, $B = \ell \times w$.

The area of the base of the pyramid is $108\ \text{cm}^2$.

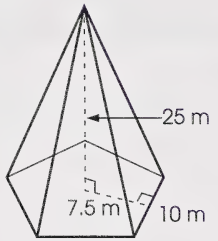
Now, substitute the area of the base into the volume formula.

$$\begin{aligned} V &= \frac{1}{3}BH \\ &= \frac{1}{3}(108)(15) \\ &= 540 \end{aligned}$$

The volume of the pyramid is $540\ \text{cm}^3$.

Example 2

Find the volume of the pyramid given.



Solution

Since $V = \frac{1}{3}BH$, you must determine B first.

$$\begin{aligned} B &= \frac{nsa}{2} \\ &= \frac{5(10)(7.5)}{2} \\ &= 187.5 \end{aligned}$$

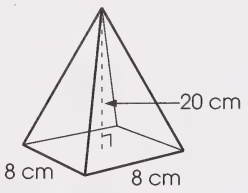
In this case, the base is a pentagon; therefore, use the formula for the area of a regular polygon.

$$\begin{aligned} \therefore\ V &= \frac{1}{3}BH \\ &= \frac{1}{3}(187.5)(25) \\ &= 1562.5 \end{aligned}$$

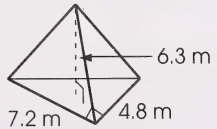
The volume of the pyramid is $1562.5\ \text{m}^3$.

4. Find the volumes of each of the following pyramids. Round your answers to the nearest tenth. A list of formulas for geometric figures is given at the back of the Appendix.

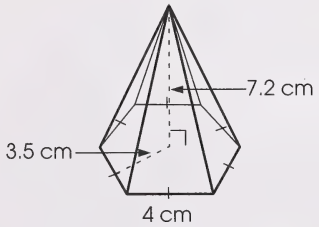
a.



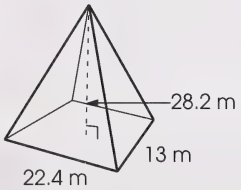
b.



c.



d.



5. The pyramids in Egypt were built in ancient times as monumental tombs for Egyptian kings. The pyramid in the photograph (built around 2700 B.C.) was the tomb of the Egyptian pharoah Khufu (or Cheops). It has a square base, with each side measuring 230 m, and an original height measuring 147 m.

a. Find the volume of stone used to construct this pyramid. Assume that the pyramid is solid stone, with no rooms or passageways.

b. The stone blocks used to build this pyramid were of varying sizes and totalled about 2.3×10^6 in number. What was the average volume of each stone block?

6. The Muttart Conservatory in Edmonton is made up of four pyramids. These pyramids house different kinds of plant life. The smaller pyramids have a square base, with each side measuring 21 m. If the volume is about $2013.9\ m^3$, what is the height of the smaller pyramids?

7. The Martins have a tent set up in their backyard that is in the shape of a regular pentagonal pyramid. The area of the floor of the tent is $11.4\ m^2$, and the height of the tent is 3.2 m. What volume of air is in the tent? Round your answer to the nearest tenth of a cubic metre $\left(m^3\right)$.

8. A cube can be divided into six congruent pyramids. Each face of the cube serves as a base for a pyramid. The edges of the cube are 60 cm long. What is the volume of each pyramid? Find your answer in two different ways.

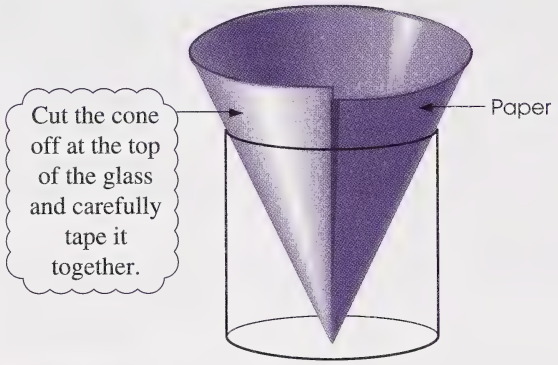
Check your answers by turning to the Appendix.

Volume of Cones

In previous mathematics courses you found a formula for the volume of a cylinder. The formula for the volume of a cylinder is $V = \pi r^2 h$. It is the area of the base of the cylinder times the height. There is a relationship between the volume of a cylinder and the volume of a cone.

Complete the following steps and answer the questions to discover this relationship.

Step 1: Use an empty glass (or some other cylindrical container) to construct a paper cone that just fits inside the glass. (Refer to the diagram.) Cut the cone appropriately and tape it together.



9. Is the radius of the top of the cone the same as the radius of the glass?

Check your answer by turning to the Appendix.

Step 2: Fill the cone with rice (or any other small-grained, pourable material); then pour it into the glass.

10. How many times must you fill the cone and pour it into the glass in order to fill the glass completely?

11. If you know the volume of the glass, how can you find the volume of the cone?

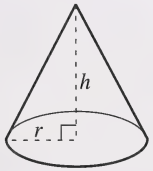
12. Write a formula for the volume of a cone based on your observations.

Check your answers by turning to the Appendix.

Since it took three full cones to fill the glass, the volume of the glass (cylinder) is exactly three times the volume of the cone with the same base and height.

The volume of a cone is one-third of the volume of a cylinder with an equal base and height.

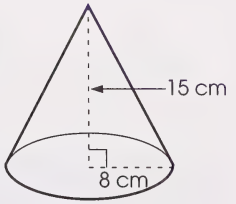
$$V = \tfrac{1}{3}\pi r^2 h$$



The formula for the volume of a cone can be used to calculate the volume of various cone-shaped figures and objects.

Example

Find the volume of the cone given. Round your answer to the nearest whole number.



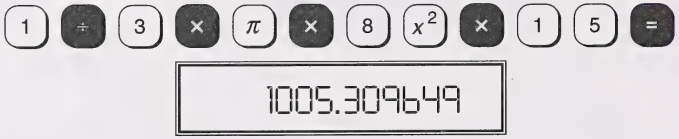
Solution

Use the formula for the volume of a cone, and substitute the given dimensions.

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi (8)^2 (15)$$
$$= \frac{1}{3}\pi (64)(15)$$
$$= 1005.309\,649$$

The volume of the cone is approximately $1005\ \text{cm}^3$.

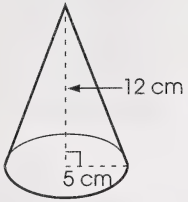
To answer the preceding example using a scientific calculator, enter the following keystroke sequence.



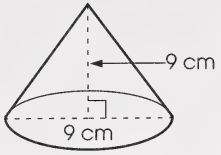
You may use a scientific calculator to answer the following questions. When necessary, round your answer to the nearest whole number.

13. Find the volume of each of the following cones.

a.



b.



c. a cone with a radius of 6 cm and a height of 14 cm

d. a cone with a diameter of 18 cm and a height of 20 cm

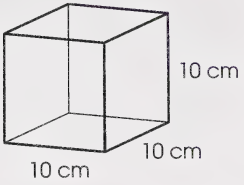
14. A cone-shaped pile of sand contains $314\ m^3$ of sand. If the pile of sand covers $78.5\ m^2$, how high is the pile of sand?

15. A snow cone has a diameter of 8 cm and a height of 12 cm. How much crushed ice can the cone hold if it is filled level to the top?

16. The cups used at a water dispenser on a golf course are cone-shaped. If the cups have a radius of 3.5 cm and a height of 7.2 cm, then how much water (in mL) can each cup hold?

(**Hint:** $1\ cm^3 = 1\ mL$)

17. The cube shown represents 1 L or 1000 mL.



a. Calculate the capacity of a cylinder that just fits inside the cube.

b. Calculate the capacity of a cone that just fits inside the cube.

c. Express as common fractions approximately what part of the cube is filled by the cylinder and what part of the cube is filled by the cone.

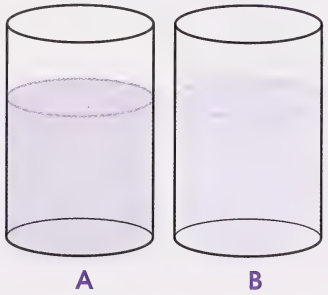
d. Which would fill more of the cube, a cone or a rectangular pyramid that just fits inside? Explain.

Check your answers by turning to the Appendix.

Now Try This

Use one of the problem-solving strategies to solve the following problem.

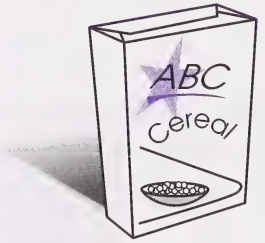
18. Container A contains three cups of water. Container B contains three cups of milk. A cup of milk is removed from container B and mixed with the water in container A. Then a cup of the mixture is removed from container A and mixed with the milk in container B. Is there now more milk in the water or more water in the milk? Explain.



Check your answer by turning to the Appendix.

In this activity you studied the relationship between the volume of a rectangular prism and the volume of a rectangular pyramid. You also examined the relationship between the volume of a cylinder and the volume of a cone.

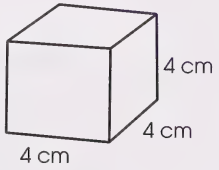
Activity 4: Volume and Surface Area in Package Design



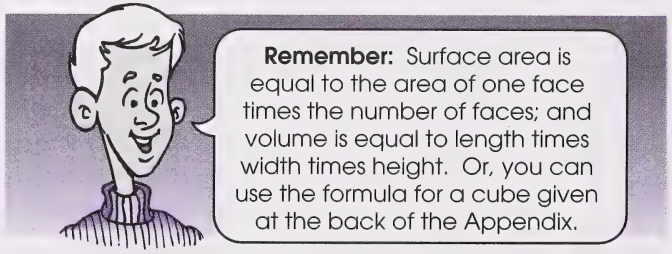
What size of cereal box does your family buy? The size of cereal boxes sold is of great importance to companies that produce and sell cereal. Surface area determines how much material (and cost) is required to produce the package, while volume indicates how much the box will hold.

In this activity you will analyse the relationship between changes in dimensions, surface area, and volume.

You are given a cube similar to the one shown on the right. Use this cube to answer questions 1 to 5.



1. Calculate the surface area and the volume of the cube.



Remember: Surface area is equal to the area of one face times the number of faces; and volume is equal to length times width times height. Or, you can use the formula for a cube given at the back of the Appendix.

2. Double all the dimensions of the cube.

   a. Calculate the surface area and the volume.

   b. How many times does the surface area increase when the dimensions are doubled?

   c. How many times does the volume increase when the dimensions are doubled?

3. Express the increases in questions 2.b. and 2.c. as powers of 2.

4. Triple all the dimensions of the cube.

   a. Calculate the surface area and the volume.

   b. How many times does the surface area increase when the dimensions are tripled?

   c. How many times does the volume increase when the dimensions are tripled?

5. Express the increases in questions 4.b. and 4.c. as powers of 3.

6. Copy and complete the following chart for changes in surface area and volume based on changes in dimensions.

Change in Dimensions	Change in Surface Area	Change in Volume
doubled	2^2	2^3
tripled		
halved		
n times		

7. Which changes faster for a given change in the dimensions, surface area or volume?

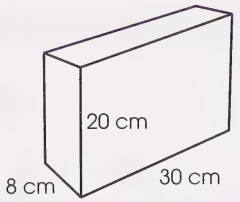
8. Write, in words, the relationship between changes in surface area and changes in dimension, and changes in volume and changes in dimension.

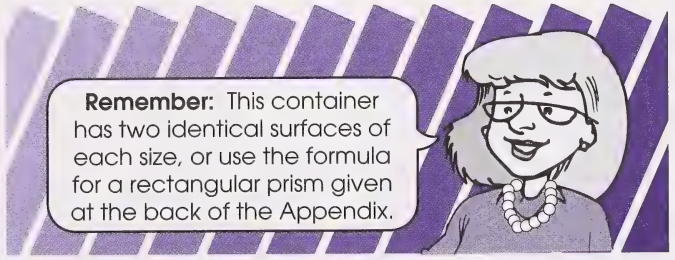
9. Explain why the exponent for change in surface area is 2, while the exponent for change in volume is 3.

Check your answers by turning to the Appendix.

Now test the relationship for a rectangular container similar to a box of cereal.

10. Find the surface area and the volume of the rectangular container on the right.





11. Double the dimensions of the container in question 10.

a. Calculate the surface area and the volume.

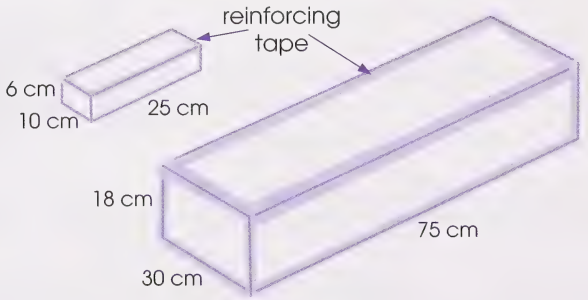
b. Is the relationship between doubling the dimensions and the change in surface area and the change in volume the same as the relationship for a cube?

12. a. What effect does doubling the dimensions of a box of cereal have on the material needed to make the box and on the amount of cereal it can hold?

b. What implications does this have for the cereal industry?

c. Why do cereal packages come in a variety of sizes?

13. ABC Manufacturing Inc. has two similar types of packages for one of their products.



a. What is the ratio of the lengths of the sides of the larger box to the lengths of the sides of the smaller box?

b. How much more reinforcing tape is needed for the larger box as compared to the smaller box?

c. What is the relationship in the amount of material needed to build each box?

d. How do the volumes of the two boxes compare?

14. A soup can, with a radius of 4 cm and a height of 12 cm, has a surface area of $402\ \text{cm}^2$ and a volume of $603\ \text{cm}^3$.

a. Predict the surface area and the volume of the can if the radius and height are doubled.

b. Test your prediction in question 14.a. Round your answers to the nearest whole number. **Remember:** The surface area includes both the top and bottom of the can, while volume is the area of one end times the height.

Check your answers by turning to the Appendix.

If you have access to a computer spreadsheet program like *ClarisWorks*™, you may wish to use it to do question 15.

If you do not have access to a spreadsheet program, use a scientific calculator to do question 15. To use your calculator efficiently, use the memory key for πr^2 since the radius of every can is the same.

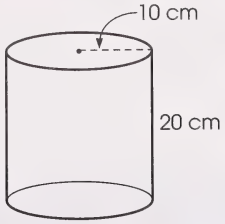
15. a. Copy and complete the following chart to find the surface area and the volume for various heights of cans with a radius of 4 cm. Round your answers to the nearest whole number.

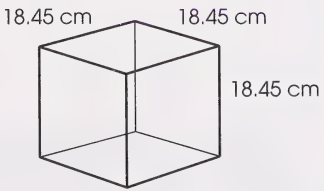
Height h (cm)	Surface Area $SA\ (\text{cm}^2)$	Volume $V\ (\text{cm}^3)$
4	201	402
5		
6		
7		
8		
12		

b. Draw the graphs of surface area versus height and volume versus height. Label each graph. How do the graphs compare?

c. Join each set of points with a straight line and extend each line back to the y-axis. What is the surface area of a can with a height of 0 cm? Why? What is the volume of a can with a height of 0 cm? Why?

16. The following two containers have the same volume.





a. Calculate the surface area of each container to the nearest square centimetre.

b. How do the surface areas compare?

c. What does this tell you about surface area versus volume for these containers with different shapes? How does this affect the packaging industry?

You may use a computer spreadsheet program to help answer question 17.

17. You have been hired by a local movie theatre to design an open container with the following specifications:

- a square base
- will hold $4000\ cm^3$ of popcorn
- will require the least amount of material to make

Hint: A popcorn container used in a movie theatre does not require a top.

a. Determine the square container that holds approximately $4000\ cm^3$ of popcorn and has the least surface area. Copy and complete a chart similar to the following using trial and error and a calculator. (Round to the nearest hundredth where necessary.)

Volume (cm^3)	Length (cm)	Width (cm)	Height (cm)	Surface Area (cm^2)
4000	10	10		

b. What shape of container might be more efficient? Experiment with various values for radius and height to determine the most efficient dimensions.

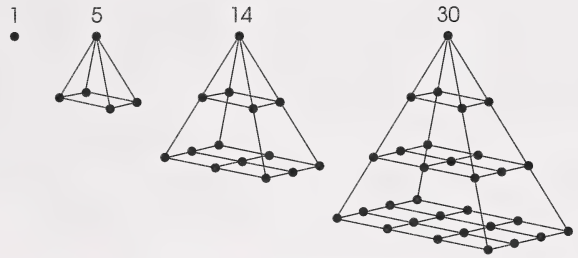
c. What factors other than minimum surface area might influence the size and shape of a container?

Check your answers by turning to the Appendix.

Now Try This

Solve the following challenge problem on pyramidal numbers.

18. You are familiar with even numbers (2, 4, 6, 8,...), odd numbers (1, 3, 5, 7,...), and square numbers (1, 4, 9, 16,...); but have you heard of pyramidal numbers? Pyramidal numbers are illustrated in the following diagram.



a. What would the next pyramidal number be?

b. What kind of numbers make up each row?

c. How would you find the tenth pyramidal number?

Check your answers by turning to the Appendix.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

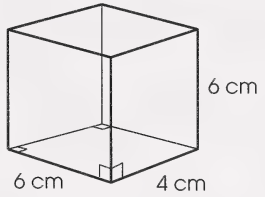
Extra Help

The volumes of pyramids and cones are equivalent to one-third of the volumes of prisms and cylinders that have the same base and height. That is how you derived the formulas for the volume of a pyramid and the volume of a cone.

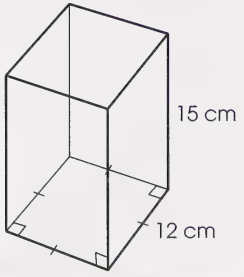
You also discovered that changes in the dimensions bring predictable changes in the volume. In particular, the factor by which the dimensions have all been changed (doubled, tripled, halved), raised to the third power, will give you the resultant change in the volume. It is to the third power because there are three dimensions. Keep these concepts in mind as you do the following questions.

1. Find the volume of the following prisms and cylinder to the nearest whole number.

a.



b.



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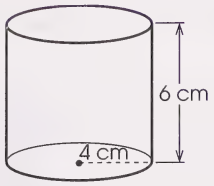
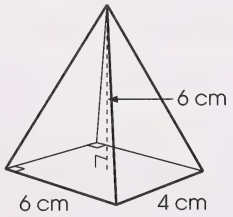
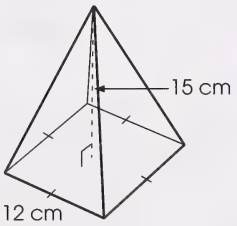


2. Find the volume of the following pyramids and cone to the nearest cubic centimetre $\left(\text{cm}^3\right)$.

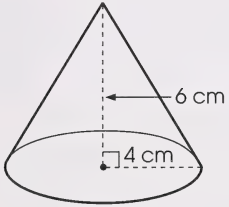
a.



b.



c.



3. Compare questions 1.a. and 2.a., 1.b. and 2.b., and 1.c. and 2.c.

a. Do you notice any similarity within each pair of figures?

b. How do their volumes compare?

4. The volume of a prism, cylinder, pyramid, or cone is given. Also given is the factor by which the dimensions have all been changed to produce a new figure. Find the volume of the new figure.

a. $V = 200\ \text{cm}^3$
Dimensions: doubled

b. $V = 48\ \text{cm}^3$
Dimensions: halved

c. $V = 30\ \text{cm}^3$
Dimensions: tripled

d. $V = 256\ \text{cm}^3$
Dimensions: quartered

Check your answers by turning to the Appendix.

Enrichment

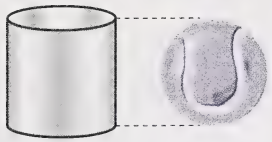
In this section you found volumes of cylinders and cones. How could you find the volume of a soccer ball or a volleyball?

You can find the volume of a sphere using a water-displacement method. (You may have done a test like this in a science class.)

For instance, in order to find the volume of a tennis ball, you need to gather the following materials; then follow the steps listed.

- an old tennis ball
- an empty 355-mL frozen juice container
- a pan to hold water
- some water

Step 1: Measure the ball's diameter and cut the container so that when the ball is placed in the container it is level with the top.

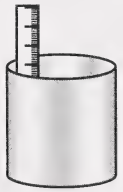


Step 2: Fill the juice container with water and place it in the pan.

Step 3: Soak the tennis ball in water; then slowly push the ball into the juice container until it is all the way in, allowing the water to overflow into the pan.

Step 4: Remove the juice container from the pan; remove the ball; and empty the juice container in the sink.

Step 5: Pour the water from the pan into the juice container, and measure the depth of the water.



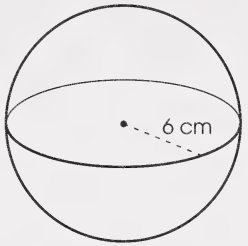
1. What fraction of the can is filled with water when you are finished?

2. The diameter of the juice container is the same as its height. The ball has the same diameter as the container. What have you learned about the volume of a sphere compared to the volume of a cylinder with the same diameter and equal height.

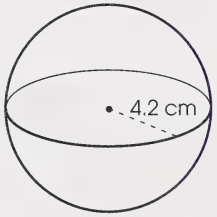
3. The cylinder and the sphere have a radius of r. Write a formula for the volume of a sphere based on the formula for the volume of a cylinder.

4. In question 3, you discovered the formula for finding the volume of a sphere. Now, find the volume of each of the following spheres. Round your answers to the nearest whole number.

a.

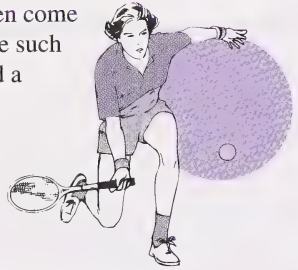


b.



5. A bowling ball is made of hardened rubber. If a ten-pin bowling ball has a radius of 10.8 cm, how much rubber does it contain (ignoring the finger holes)? Round your answer to the nearest whole number.

6. When you buy tennis balls they often come three to a cylindrical container. One such container has a radius of 3.9 cm and a height of 23.4 cm.



a. What fraction of the space inside the container do you predict the balls fill?

b. Calculate the volume of the container and the total volume of the three tennis balls. Round your answer to the nearest whole number.

c. Find the ratio of the volume of the balls to the volume of the container, and see if your prediction is correct. Round your answer to two decimal places.

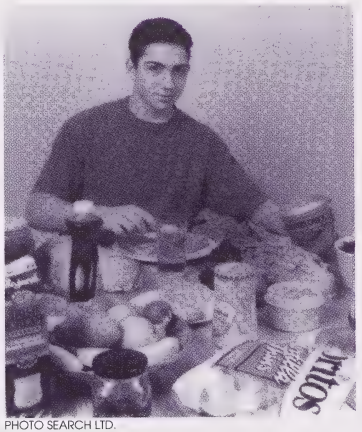
7. A balloon, in the shape of a sphere, has a radius of 10 cm.

a. If the balloon is blown up so that it has a radius of 20 cm, predict how many times its volume has increased.

b. Calculate the volumes, and check your prediction in question 7.a. Round the volumes to the nearest whole number.

Check your answers by turning to the Appendix.

Conclusion



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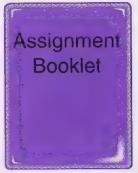
Drawing and visualizing in two and three dimensions are important skills in drafting and architecture. In this section you created figures in both two dimensions and three dimensions. You drew two-dimensional views of three-dimensional models and used visualization to transpose those two-dimensional drawings into a three-dimensional drawing or model. You also calculated the volume of pyramids and cones and analysed the importance of dimension changes in the packaging industry. You then used your improved visualization skills to design containers that could be used in the packaging industry.

The next time you buy a package of cereal or a can of soup, take a look at it and try to determine why it is a particular size and shape and what factors may have been used to design that package.

Assignment



You are now ready to complete the assignment for Section 2.

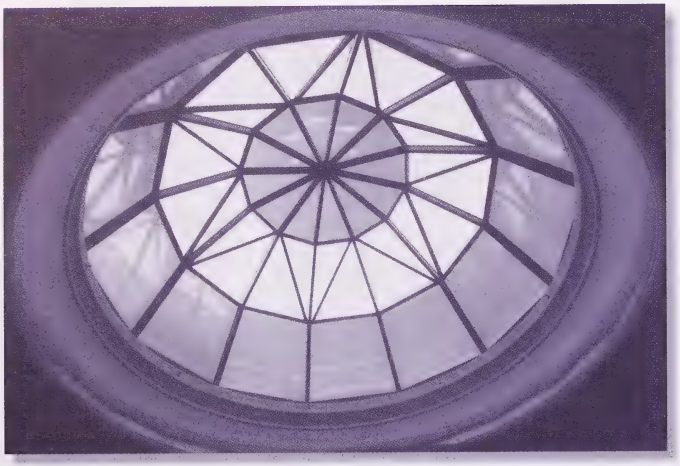
Section 3: Triangles and Trigonometry

How high is the tower in the photograph? Can you find the height of this tower without actually climbing it or in some way running a tape measure from the top to the ground?

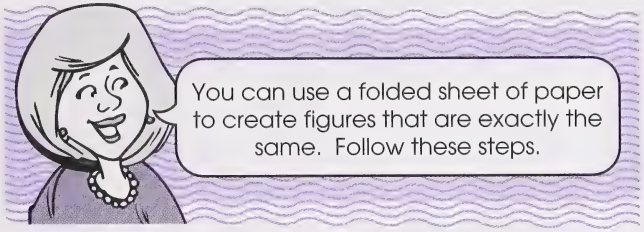
You can if you use a method of measurement called indirect measurement. Mathematicians, engineers, navigators, and many other people have been using indirect measurement of some kind for thousands of years. You had some experience with indirect measurement when you studied the theorem of Pythagoras in previous mathematics courses.

In this section you will begin with some basic properties of congruent and similar triangles, which will lead to the development of the primary trigonometric ratios. Trigonometric ratios are the basis of the area of mathematics called trigonometry. Trigonometry deals with the relationship between angles in a right triangle and the ratio of their sides. You will discover how to find the measures of the missing angles and sides in a right triangle and then use your knowledge of trigonometric ratios in solving practical problems.

Activity 1: Congruent Triangles



Can you find triangles that appear to be exactly the same in the photograph? In this activity you will analyse triangles that are the same size and shape and discover the conditions that are necessary for triangles to be congruent.



Step 1: Fold a sheet of paper in half twice in the same direction.

Step 2: Draw half of a figure (such as an arrow along the folded edge of the paper), and cut out the figure.

1. Compare the two figures. Do they coincide exactly when you place one on the other?

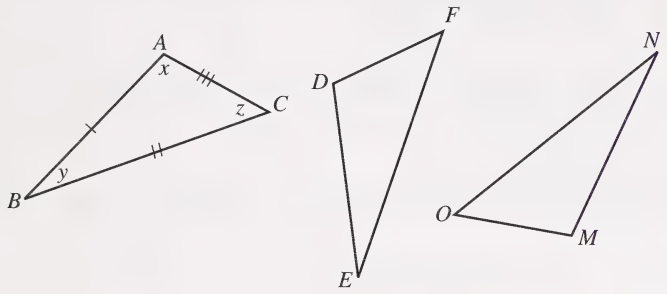
2. Do the two figures appear to be the same size and shape?

3. Suppose you were to fold the paper three times along its length. How many figures would you get? Do all these figures appear to be the same size and shape?

Check your answers by turning to the Appendix.

Figures that are identical in size and shape are called **congruent figures**.

All of the following triangles have the same size and shape.



4. Copy or trace the preceding triangles on a piece of paper; then cut them out. Use these triangles to answer the following questions.

   a. Find the sides that are the same length as AB in each of the other two triangles. Mark them similarly.

   b. Use two ticks to mark the sides that are the same length as BC in the other two triangles.

   c. Mark the remaining unmarked sides of the two triangles with three ticks to show that they are the same length as AC.

   d. Find the angles that are the same as angles A, B, and C in the other two triangles. Mark them with x, y, and z.

   e. Place the three triangles one on top of the other. Do the three triangles coincide exactly?

Check your answers by turning to the Appendix.

Since the triangles in question 4 are the same size and shape, you can say they are congruent. That is, $\Delta ABC \cong \Delta DEF \cong \Delta MNO$.

The symbol $\cong$ means "is congruent to."

Note: The vertices of the triangles are always listed in the order of their corresponding angles.

$\Delta ABC \cong \Delta DEF \cong \Delta MNO$ means that the pairs of corresponding angles and corresponding sides are equal.

Angles	Sides
$\angle A = \angle D = \angle M$	$AB = DE = MN$
$\angle B = \angle E = \angle N$	$BC = EF = NO$
$\angle C = \angle F = \angle O$	$AC = DF = MO$

These six statements can always be made about any two congruent triangles.

It is not necessary to know that all of the pairs of corresponding sides and corresponding angles are equal in order to show that two triangles are congruent. Now you will see if you can discover how much information is necessary to show or draw two or more congruent triangles.

5. Use a ruler, protractor, and compass to accurately draw three triangles for each of the following properties. Label each triangle with the property number, and then cut them out. You will end up with 15 triangles.

Property 1: The sides are 4 cm, 6 cm, and 8 cm in length.

Property 2: The angles are 40°, 60°, and 80°.

Property 3: Two angles are 50° and 60° and the side joining these two angles is 7 cm.

Property 4: One angle is 80° and the two sides that include the given angle are 4 cm and 6 cm long.

Property 5: Two sides are 6 cm and 5 cm and one of the angles not formed by the given sides is 50°.

6. Look at the triangles in each group from question 5 according to their property numbers. State whether all possible triangles in each group are congruent.

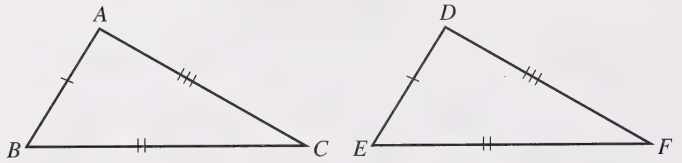
7. Based on your observations, how much information is necessary to draw two or more congruent triangles?

Check your answers by turning to the Appendix.

There are three necessary conditions under which two triangles are congruent.

Condition 1: SSS (side, side, side)

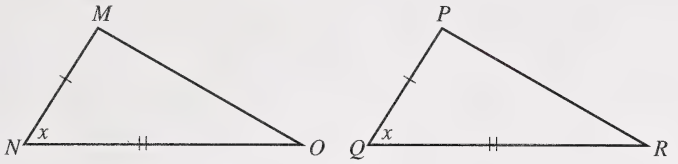
All three sides of one triangle are equal to the corresponding sides of the other triangle.



$\Delta ABC \cong \Delta DEF$

Condition 2: SAS (side, angle, side)

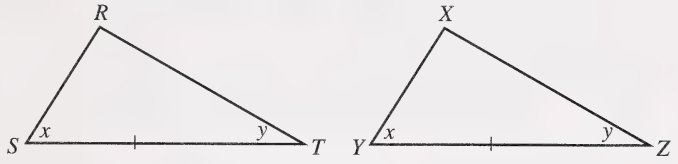
Two sides and the contained angle of one triangle are equal to the corresponding two sides and the contained angle of the other triangle.



$\Delta MNO \cong \Delta PQR$

Condition 3: ASA (angle, side, angle)

Two angles and a side of one triangle are equal to the corresponding two angles and a side of the other triangle.



$\Delta RST \cong \Delta XYZ$

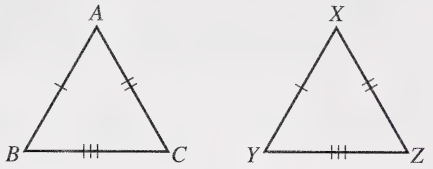
Note: It does not have to be the contained side of the two angles.

$\angle R = \angle X$ $\qquad$ $180 - (x + y)$

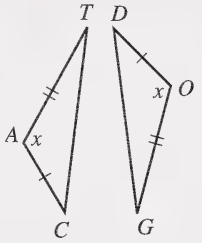
Therefore, Condition 3 can be referred to as AAS or SAA as well.

8. The following pairs of triangles are congruent. Write the congruency statement that indicates the conditions under which the two triangles are congruent.

a.



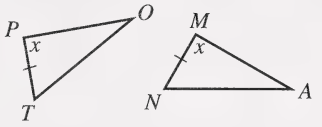
b.



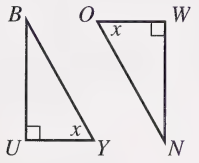
9. List all the corresponding congruent parts of the congruent triangles in question 8.

10. Name the missing congruent angles or congruent sides that are needed to show that the following pairs of triangles are congruent.

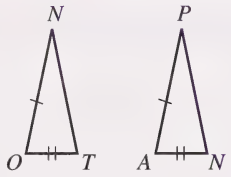
a.



b.

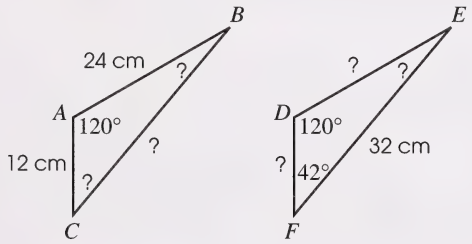


c.

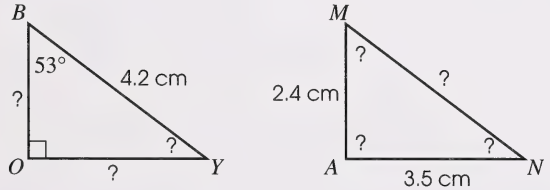


11. Given that the following pairs of triangles are congruent, find the missing measures (indicated by the ?).

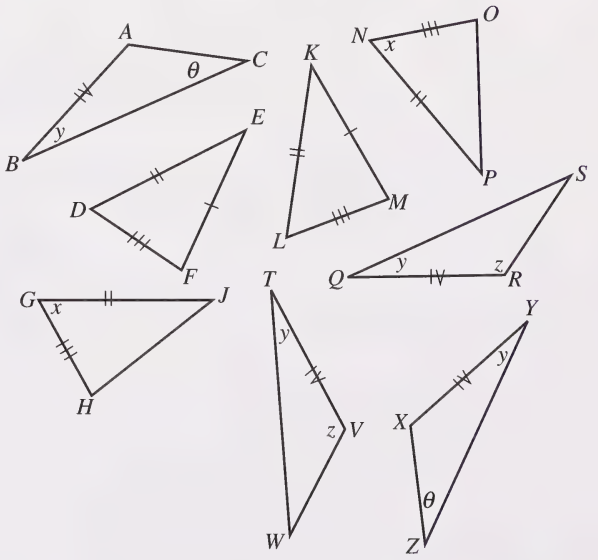
a. $\Delta ABC \cong \Delta DEF$



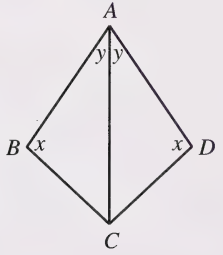
b. $\Delta BOY \cong \Delta MAN$



12. Using the indicated measures only, find each pair of congruent triangles in the following diagram. Write the congruence statement for each pair of triangles, and indicate which congruency condition is met for each pair.



13. Use the diagram on the right to answer the following questions.

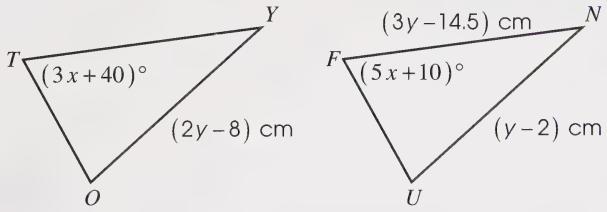


a. Why does $\angle ACB = \angle ACD$?

b. Are the two triangles congruent? Explain.

14. $\Delta PIN \cong \Delta MAT$. Name the corresponding congruent parts.

15. $\Delta TOY \cong \Delta FUN$



a. What is the measure of $\angle T$?

b. What is the measure of OY?

c. What is the measure of TY?

16. a. Are two triangles with the same perimeter always congruent, sometimes congruent, or never congruent?

b. Are two triangles with the same area always congruent, sometimes congruent, or never congruent?

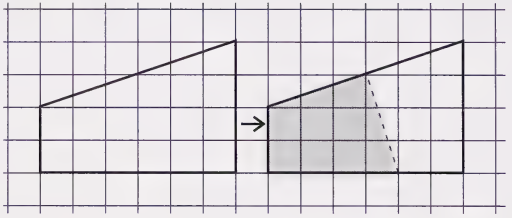
c. Are two squares with the same perimeter always congruent? Are two squares with the same area always congruent? Why is this true?

Check your answers by turning to the Appendix.

Now Try This

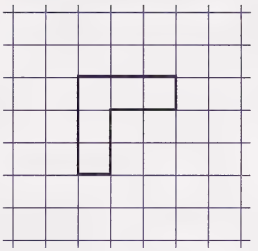
Use one of the problem-solving strategies to solve the following problem.

Some shapes can be divided into two congruent parts. The following is an example.

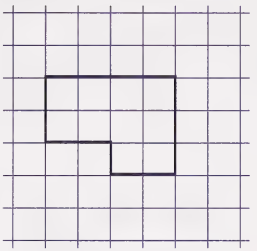


17. Draw the following shapes on some graph paper; then divide each one into two congruent parts.

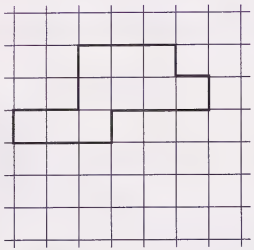
a.



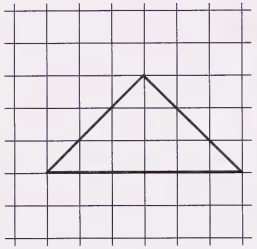
b.



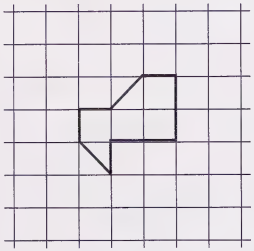
c.



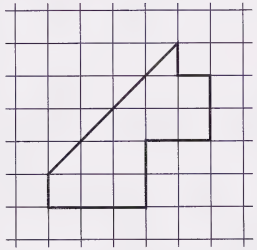
d.



e.



f.



18. State which transformation is required to move one congruent part onto the other in each of questions 17.a. to 17.g.

Check your answers by turning to the Appendix.

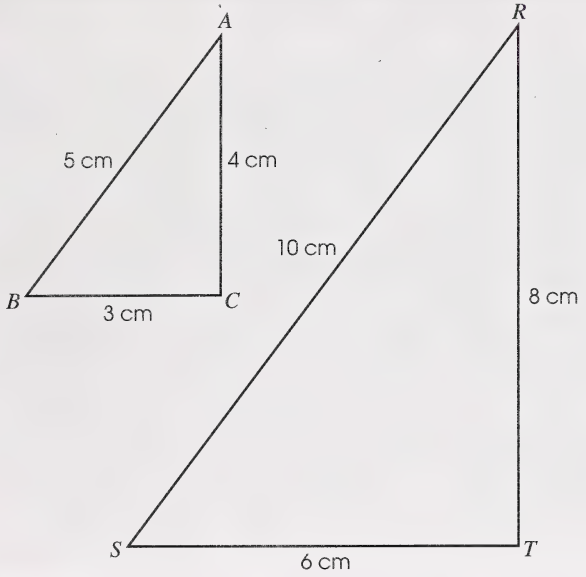
You examined how to recognize when two triangles are congruent, and have discovered what conditions are sufficient in order to describe two triangles as being congruent.

Activity 2: Similar Triangles

Do the two logos have the same shape and size? In Activity 1 you discovered that figures that have the same shape and size are called congruent figures. In this activity you will study figures that have the same shape but not the same size. These are called **similar figures**.

Triangles can be similar as well. Just as with congruent triangles, there are certain conditions that are necessary for similar triangles. In the following exercises you will discover the conditions that are necessary for similar triangles.

Use the following two triangles to answer questions 1 and 2.



1. Measure the angles of ΔABC and ΔRST. Are the corresponding angles equal?

2. **a.** Find the following pairs of ratios.

- $\frac{AC}{AB}, \frac{RT}{RS}$
- $\frac{BC}{AB}, \frac{ST}{RS}$
- $\frac{BC}{AC}, \frac{ST}{RT}$

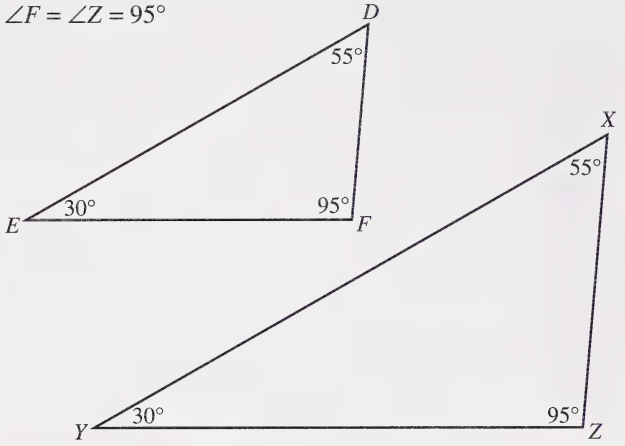
b. What can you conclude about the ratio of the lengths of any two sides of ΔABC compared to the ratio of the lengths of the two corresponding sides of ΔRST?

3. Use the following two triangles to answer the following questions.

$\Delta DEF \sim \Delta XYZ$

The symbol ~ means "is similar to."

$\angle D = \angle X = 55°$
$\angle E = \angle Y = 30°$
$\angle F = \angle Z = 95°$



a. Measure the corresponding sides of ΔDEF and ΔXYZ, and write them as ratios in simplest form. Are these ratios equal?

b. Find these pairs of ratios.

i. $\frac{DE}{DF}, \frac{XY}{XZ}$

ii. $\frac{DF}{EF}, \frac{XZ}{YZ}$

c. What do you notice about the ratios in question 3.b.? Explain.

4. In ΔABC, $\angle A = 40°$ and $\angle B = 60°$. In ΔDEF, $\angle D = 40°$ and $\angle E = 60°$.

a. What are the measures of $\angle C$ and $\angle F$? Explain.

b. Is $\Delta ABC \sim \Delta DEF$? Explain.

Check your answers by turning to the Appendix.

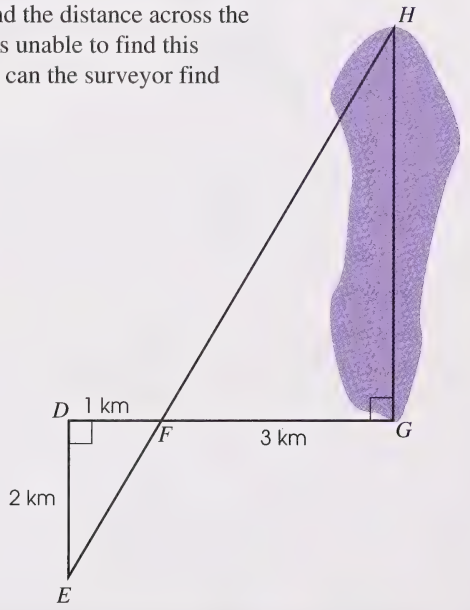
Properties of Similar Triangles

- If the corresponding sides of two triangles are proportional (have the same ratio), then the triangles are similar.
- If the ratio of any two sides of one triangle is equal to the ratio of the two corresponding sides of the other triangle, then the triangles are similar.
- If two pairs of corresponding angles of two triangles are equal, then the triangles are similar.

These facts about similar triangles are very useful in finding measures of distances that are not easily measured directly. Surveyors use these techniques in their jobs. They use an instrument called a transit to measure angles and distances.

Example 1

A surveyor wants to find the distance across the length of the lake, but is unable to find this distance directly. How can the surveyor find the distance?



Solution

Using surveying equipment, the surveyor can set up similar triangles.

$\Delta FGH \sim \Delta FDE$

$\angle D = \angle G = 90°$
$\angle DFE = \angle GFH$ (opposite angles)
$\therefore\ \angle E = \angle H$

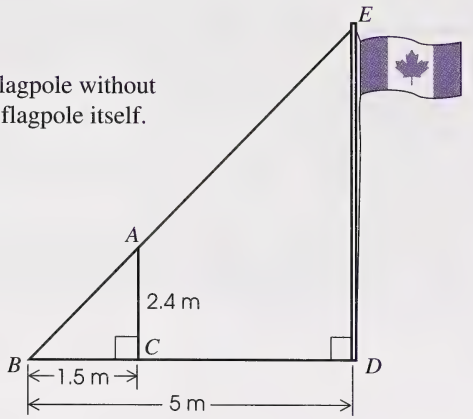
Since the two triangles are similar, the ratios of corresponding sides are equal.

$$\frac{DE}{GH} = \frac{DF}{GF}$$
$$\frac{2}{GH} = \frac{1}{3}$$
$$2 \times 3 = 1 \times GH$$
$$6 = GH$$
$$GH = 6$$

The distance across the lake is 6 km.

Example 2

Find the height of the flagpole without actually measuring the flagpole itself.



Solution

$\Delta ABC \sim \Delta EBD$

Therefore, the corresponding sides have the same ratio.

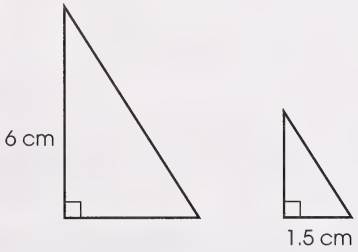
$$\frac{AC}{ED} = \frac{BC}{BD}$$
$$\frac{2.4}{ED} = \frac{1.5}{5}$$
$$2.4 \times 5 = 1.5 \times ED$$
$$12 = 1.5\,ED$$
$$1.5\,ED = 12$$
$$\frac{1.5\,ED}{1.5} = \frac{12}{1.5}$$
$$ED = 8$$

The flagpole is 8 m tall.

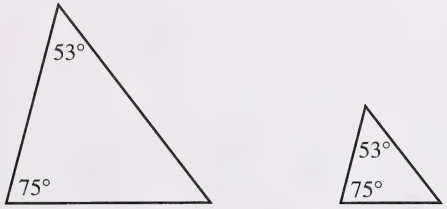
5. Determine whether each of the following pairs of triangles is similar. Use the measurements given about the triangles to help you determine your answer. Explain.

a.

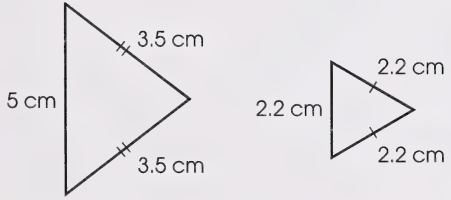
b.



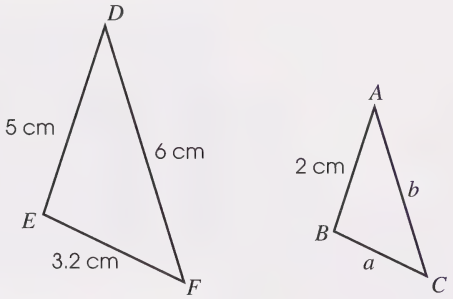
c.



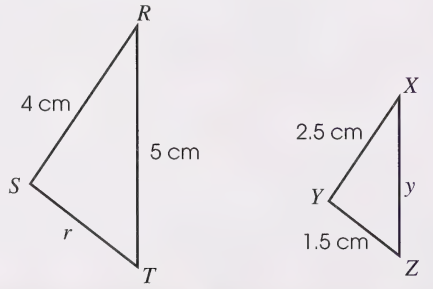
d.

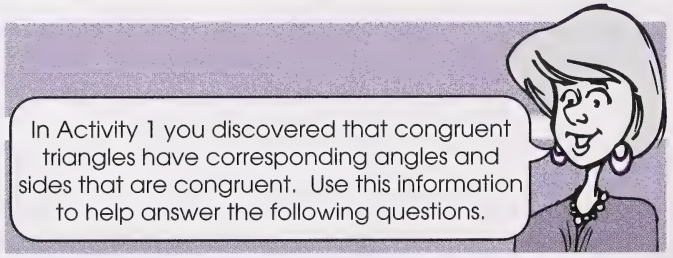


6. ΔABC and ΔDEF are similar. Find a and b.



7. ΔRST and ΔXYZ are similar. Find r and y.

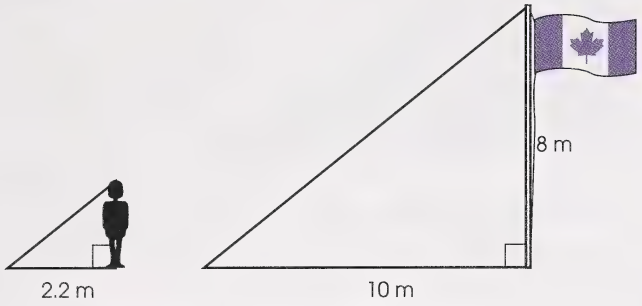




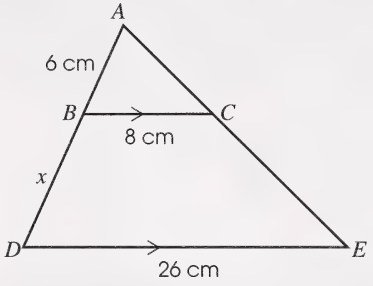
8. Are congruent triangles always similar? Explain.

9. Are similar triangles always congruent? Explain.

10. An 8-m flagpole casts a 10-m shadow. At the same time of day, Joanna casts a 2.2 m shadow. How tall is Joanna?

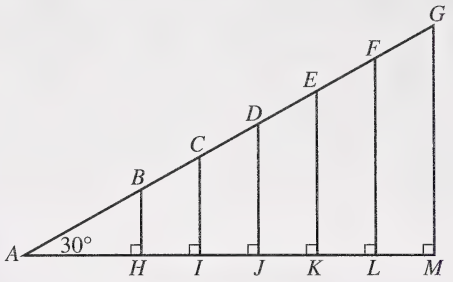


11. Find the measure of x.



12. Are all squares similar? Explain.

13. The diagram given is a group of similar triangles. Use the diagram to answer the questions that follow.



a. $\Delta ABH \sim \Delta ACI \sim \Delta ADJ$. Why?

b. What other triangles is ΔABH similar to?

c. Measure BH and AB. What is the ratio $\frac{BH}{AB}$?

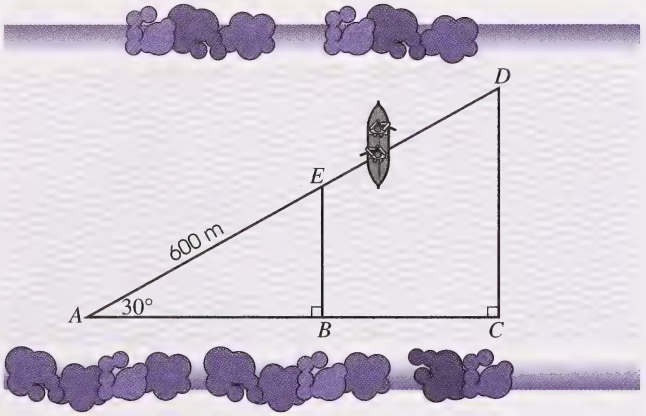
d. Measure CI and AC. What is the ratio $\frac{CI}{AC}$?

e. Predict what the ratios of $\frac{DJ}{AD}$, $\frac{EK}{AE}$, $\frac{FL}{AF}$, and $\frac{GM}{AG}$ are going to be. Explain.

f. If you extended AG to X and AM to Y so that $XY \perp AY$ and $XY = 9.5$ cm, what is the measure of AX?

g. What conclusion can you draw about a right triangle with a 30° angle?

14. Fred and Maurice are canoeing across the North Saskatchewan River. The river is flowing at a very fast rate; so, they are unable to paddle straight across. In fact, their path makes a 30° angle with the riverbank that they are paddling from.



a. When they are halfway across the river they have travelled 600 m. How far from the riverbank are they?

b. How wide is the river where they are crossing?

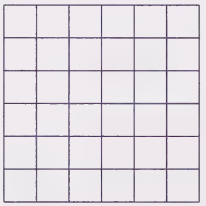
c. If Fred and Maurice wanted to land at a certain spot on the opposite riverbank, about how far upstream should they begin their crossing?

Check your answers by turning to the Appendix.

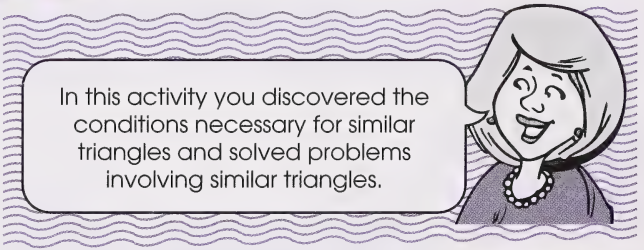
Now Try This

Use one of the problem-solving strategies to solve the following problem.

15. Copy and cut out a 6×6 square grid like the one shown. Cut this grid into exactly twelve squares so that the whole grid is cut up and no pieces remain unused. You can only cut along the horizontal and vertical lines. Get as many different solutions as you can.

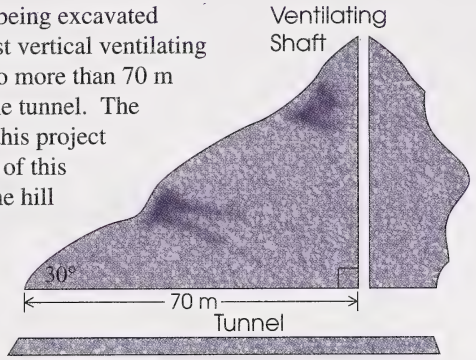


Check your answer by turning to the Appendix.



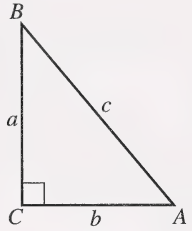
Activity 3: Trigonometry

A horizontal tunnel is being excavated through a hill. The first vertical ventilating shaft must be drilled no more than 70 m from the entrance of the tunnel. The engineers involved in this project need to find the height of this shaft and how far up the hill the shaft will exit.



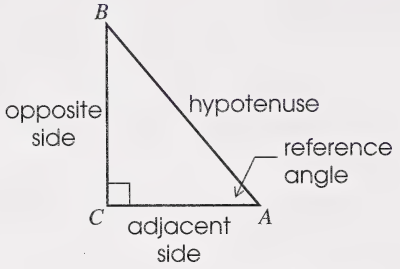
To answer these questions, the engineers need to use a branch of mathematics called trigonometry. Trigonometry involves the study of the relationships of angles and sides of right triangles.

To discuss and clearly understand trigonometry, you must learn some basic vocabulary first. As you may recall, from your previous work with the theorem of Pythagoras, the sides of a triangle can be named by using a small letter which corresponds to the letter representing the opposite angle of that side.



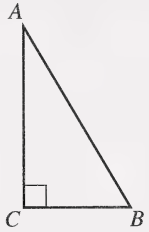
A right triangle has one right angle and two acute angles. The side opposite the right angle is the **hypotenuse**. The other two sides of the triangle have names as well; but they are named according to the position of the reference angle. One of the acute angles is chosen as the reference angle.

The side directly opposite the reference angle is called the **opposite side**. The side next to the reference angle, that is not the hypotenuse, is called the **adjacent side**.

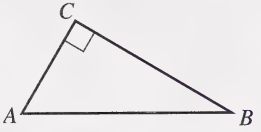


1. Name the opposite side, adjacent side, and hypotenuse with reference to $\angle A$ in each of the following right triangles.

a.

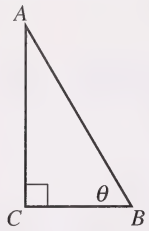


b.

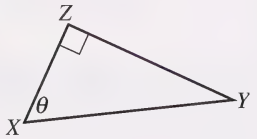


2. Name the sides appropriately with reference to $\angle\theta$ in each of the following right triangles.

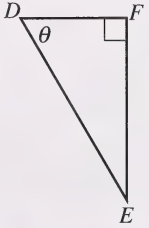
a.



b.



c.



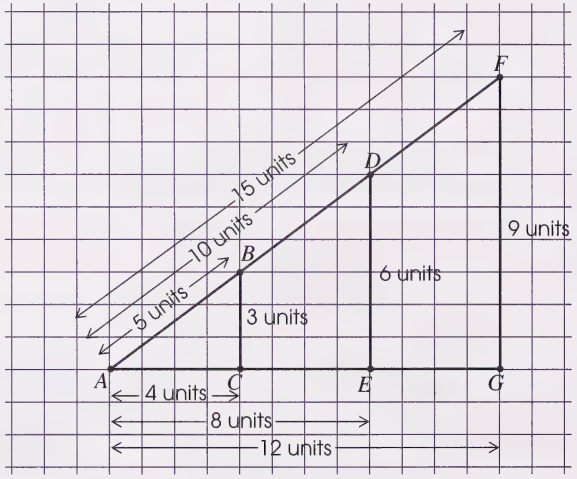
Check your answers by turning to the Appendix.

You will now use the properties of similar triangles, which you developed in Activity 2, to develop the trigonometric ratios.

Tangent Ratio

Obtain a sheet of grid paper and draw a right triangle with a base length of 4 units and a height of 3 units. Extend the triangle to make one with a base length of 8 units and a height of 6 units, and another with a base length of 12 units and a height of 9 units.

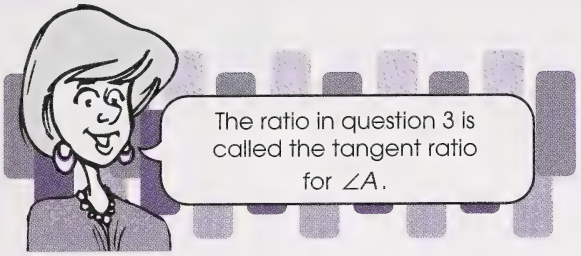
Your drawing should look similar to the following.



3. Using $\angle A$ as the reference angle, find the ratio of the opposite side (BC) to the adjacent side (AC). Do the same for ΔADE and ΔAFG. Find the decimal value for the ratio for each triangle.

4. Is the ratio in question 3 the same for each triangle?

Check your answers by turning to the Appendix.



If $\angle A$ is an acute angle in a right triangle, then

$$\tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent } \angle A}.$$

Often you will see the tangent ratio for a given angle A written in short form as $\tan A = \frac{\text{opp}}{\text{adj}}$.

5. Use a protractor to find the measure of $\angle A$ on the previous graph to the nearest degree.

Check your answer by turning to the Appendix.

You can use a scientific calculator to find the ratio of an angle. Perform the following keystroke sequence.

tan 3 7 =

Some calculators may require that you start with the number first. Consult your manual for order of operation.

6. What value appears in the display? Round this value to four decimal places.

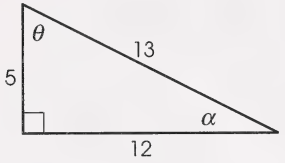
7. How does this value compare to the tangent ratio you obtained previously?

8. You can use a scientific calculator to find the tangent ratio of any angle. Find the tangent ratio of the following angles using a scientific calculator. Round your answer to four decimal places.

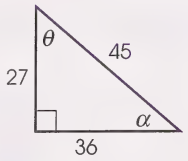
a. 81° **b.** 22° **c.** 45° **d.** 2°

9. Using the values provided, find the tangent ratios for θ and α. Express your answers as fractions (in simplest form) and as decimals. Round your answer to four decimal places.

a.



b.



Check your answers by turning to the Appendix.

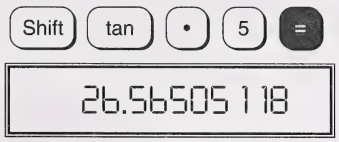
You can also use a scientific calculator to find the angle given the tangent ratio. Work through Example 1.

Example 1

If $\tan A = 0.5000$, then find $\angle A$. Round your answer to the nearest degree.

Solution

To find the measure of $\angle A$, enter the following keystrokes:



Your calculator may require you to use INV tan. Consult the owner's manual.

$\therefore \angle A \doteq 27°$

Use a scientific calculator to answer questions 10 and 11.

10. Find the angle of the following tangent ratios to the nearest degree.

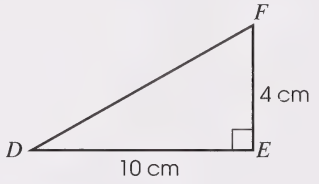
a. $\tan \theta = 0.3985$

b. $\tan \theta = 9.8476$

c. $\tan \theta = \frac{5}{7}$

d. $\tan \theta = \frac{4}{3}$

11. Find the tangent ratio and the measure of each of acute angle in ΔDEF.

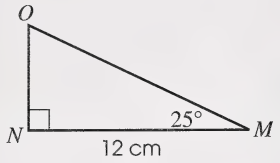


Check your answers by turning to the Appendix.

You can use the tangent ratio to solve for the length of a side in a right triangle if you know one acute angle and one of the two shorter sides.

Example 2

In ΔMNO, find the length of NO to the nearest tenth of a centimetre.

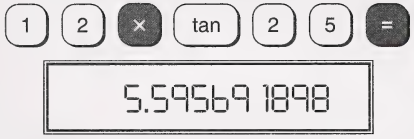


Solution

$$\tan M = \frac{\text{opp}}{\text{adj}}$$

$$\tan 25° = \frac{NO}{12}$$

$$NO = 12 \times \tan 25°$$



Therefore, the length of NO is about 5.6 cm.

12. Use the information in ΔXYZ to answer the questions that follow.

a. Calculate the length of XY to the nearest tenth of a metre.

b. Find the tangent of 65° to four decimal places.

c. Calculate $\angle X$ using two methods.

Check your answers by turning to the Appendix.

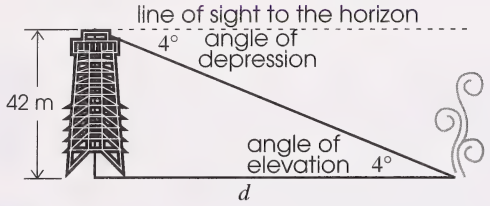
Trigonometry was initially developed for use in solving problems in navigation. Since then its application has been broadened to include a wide range of problems in mathematics, science, and industry. The tangent ratio is used with many problems involving heights.

Example 3

A forest ranger is standing at the top of his lookout tower. He notices some smoke rising from the forest. Using an instrument called a transit, he finds the angle of depression for the source of the smoke to be 4°. If the tower is 42 m high, how far from the lookout tower is the source of the smoke? Round your answer to the nearest metre.

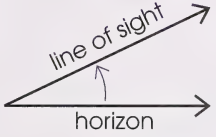
Solution

Draw a diagram of the situation.

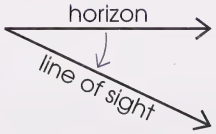


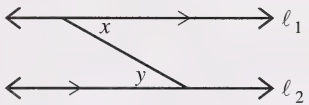
The **angle of depression** is the angle formed between the line to the horizon and the line of sight to the object which is below the observer's eye. The **angle of elevation** is the angle formed by the horizontal plane and the line of sight to the object which is above the observer's eye.

Angle of Elevation



Angle of Depression





If $\ell_1 \| \ell_2$, then $\angle x = \angle y$.
(alternate interior angles between parallel lines)

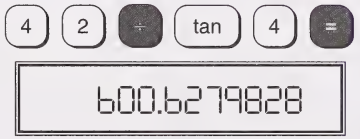
Thus, the angle of depression is equal to its corresponding angle of elevation.

Use the correct trigonometric ratio to solve for d, the distance to the smoke.

$$\tan 4° = \frac{42}{d}$$

$$d \times \tan 4° = 42$$

$$d = \frac{42}{\tan 4°}$$

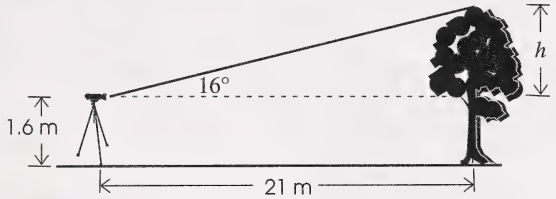


Therefore, the smoke is about 601 m from the base of the tower.

13. A tree casts a shadow 38 m long when the sun's rays are at an angle of 42° to the ground. Find the height of the tree to the nearest metre.

14. A sailboat is sighted from a lighthouse 125 m above the level of the water. If the angle of depression made with the highest tip of the sailboat is 6° and the tip of the sailboat is 10 m above the level of the water, how far is the sailboat from shore? Round your answer to the nearest metre.

15. Calculate the height of the tree to the nearest tenth of a metre, using the indicated measurements.



Check your answers by turning to the Appendix.

There are many types of instruments used for determining distances and measuring angles of elevation and angles of depression.

A transit is an instrument used by surveyors for measuring angles and determining distances. A sextant is an angular measuring device used by navigators at sea. A clinometer is an instrument used for measuring angles of inclination or slope. You can make your own clinometer.

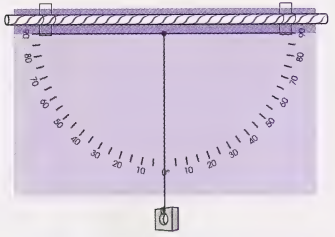
Constructing and Using a Clinometer

Materials

- a thin drinking straw or pencil
- a piece of thread or string 20 cm long
- a weight (such as a nut)
- adhesive tape and glue
- the cut-out located in the back of the Appendix
- piece of cardboard

Construction

- Cut out the clinometer located at the back of the Appendix and glue it on a piece of cardboard of the same size.
- Tape the drinking straw where indicated.
- Tie the nut to the thread and place the thread on the centre point so it hangs down through 0° and about 4 cm below the cardboard. Tape it securely on the back with adhesive tape.



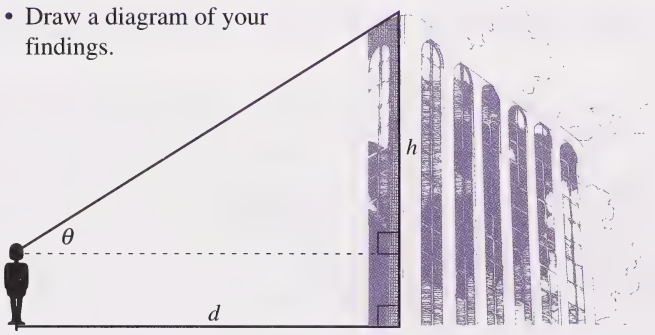
How To Use a Clinometer

- Point the straw at the top of the object you wish to measure. Look along the straw and make sure it is aligned as closely as possible.
- Have someone read the angle of the string. If you have to do it yourself, hold the string with your thumb once you have it properly aligned; then read the angle afterward. You will have to estimate to the nearest degree because the angles have only been marked at every 5°.
- This gives you the angle of elevation or depression.

Practice

Practise using your clinometer by finding the height of a nearby building.

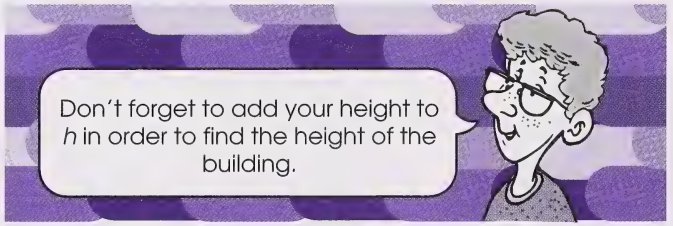
- Stand with your clinometer where you can see the top of the building.
- Use your clinometer to measure the angle of elevation.
- Measure the distance (d) from where you are standing to the base of the building.
- Draw a diagram of your findings.



- Set up your equation using the correct trigonometric ratio.

$$\tan \theta = \frac{h}{d}$$

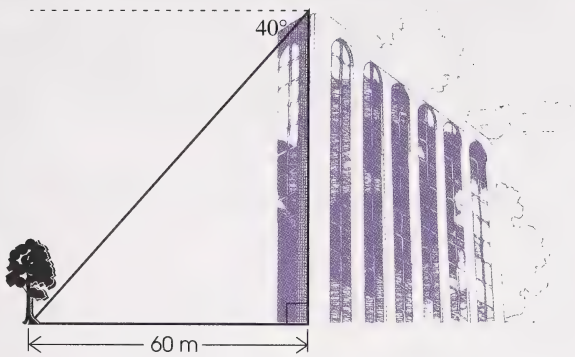
- Rearrange the equation, substitute your value for d, and solve for h.



16. Use the clinometer to find the height of some building or object that is easy for you to find the actual height. Do you think your result is reasonable?

17. Sharon used a clinometer and a tape measure to find the approximate height of a hot-air balloon that was just preparing for a flight. The angle of elevation was 20° and the distance to the base of the balloon was 80 m. What did she find the height of the inflated hot-air balloon to be if her height is 1.7 m? Round your answer to the nearest tenth of a metre.

18. From the top of a building, the angle of depression to the base of a tree 60 m away is 40° (as measured by a clinometer). Find the height of the building to one decimal place. **Remember:** the angle of depression is equal to the angle of elevation.



19. By using sonar, the captain of a fishing schooner detects a school of fish at the depth of 58.4 m. If the angle of depression is 16°, how far must the captain move the schooner in order to be directly over the school of fish? Round your answer to the nearest metre.

20. **a.** Use the tangent key on your calculator to complete the following table. Round the tangent values to four decimal places before writing them in the table.

Angle	Tangent	Angle	Tangent
5°		50°	
10°		55°	
15°		60°	
20°		65°	
25°		70°	
30°		75°	
35°		80°	
40°		85°	
45°			

b. What pattern do you notice in the value of the tangent ratio as the angle increases from 5° to 85°?

c. What do you think the tangent value for 0° will be? Test your prediction with your calculator.

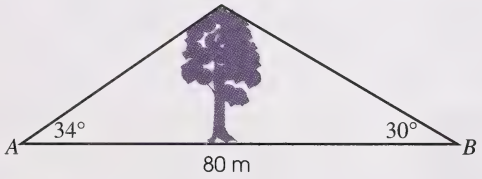
d. What do you think the tangent value for 90° will be? Test your prediction with your calculator. Why do you think this happens?

Check your answers by turning to the Appendix.

Now Try This

Solve the following challenge problem.

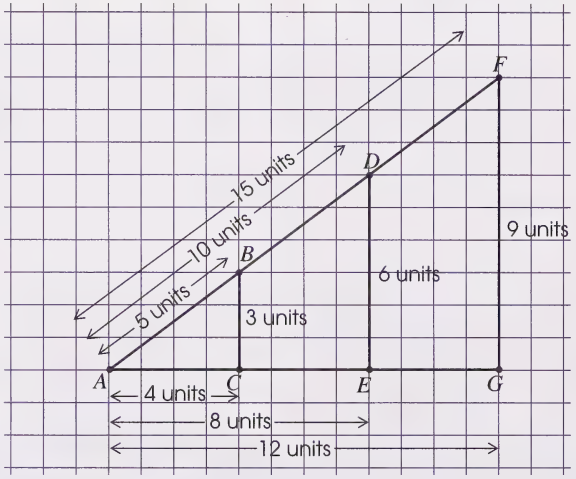
21. Using the following diagram, calculate the height of the tree to the nearest metre. [**Hint:** Let the distance from A to the tree be n. Then find the distance from B to the tree in terms of the distance of AB and n. Express the height of the tree (h) using two different trigonometric ratios and then equate them.]



Check your answer by turning to the Appendix.

Sine and Cosine Ratios

Look at the triangles you drew on the grid at the start of this activity again.

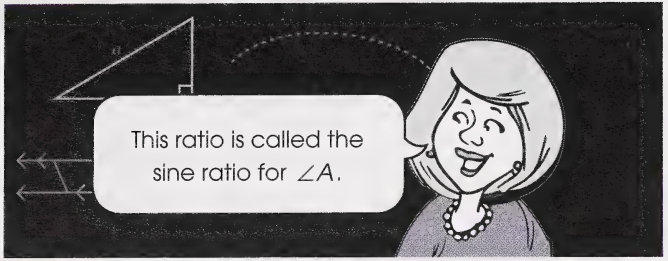


Every triangle has three sides, so there are two additional ratios with reference to $\angle A$.

22. **a.** Using $\angle A$ as the reference angle, find the ratio of the opposite side (BC) to the hypotenuse (AB). Do the same for ΔADE and ΔAFG. Use your calculator to find the decimal value for the ratio for each triangle.

b. Are the ratios the same for each triangle?

Check your answers by turning to the Appendix.



If $\angle A$ is an acute angle in a right triangle, then

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}}.$$

Often you will see the sine ratio of a given angle A written in short form as $\sin A = \frac{\text{opp}}{\text{hyp}}$.

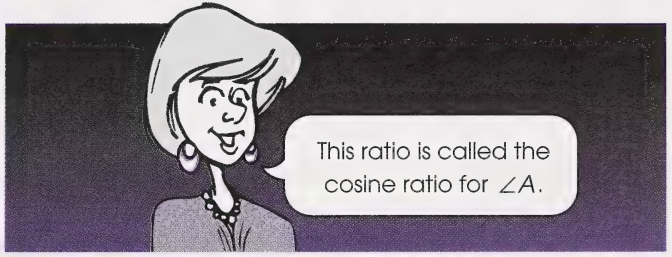
Since $\angle A$ is the same angle you used when you found the tangent ratio, its measure must be approximately 37°. Use your calculator to find angle A from the sine ratio. Enter the following keystrokes on your calculator.

23. What value appears in the display? Round the value to the nearest degree. What angle is this value?

24. **a.** Using $\angle A$ as the reference angle, find the ratio of the adjacent side (AC) to the hypotenuse (AB). Do the same for ΔADE and ΔAFG. Use your calculator to find the decimal value for the ratio for each triangle.

b. Are the ratios the same for each triangle?

Check your answers by turning to the Appendix.



If $\angle A$ is an acute angle in a right triangle, then

$$\cos A = \frac{\text{length of side adjacent } \angle A}{\text{length of hypotenuse}}.$$

Often you will see the cosine ratio of a given angle A written in short form as $\cos A = \frac{\text{adj}}{\text{hyp}}$.

25. Use a scientific calculator to find $\angle A$ from the cosine ratio. Write out the required keystrokes and the resulting display. Round the result to the nearest degree.

26. Is the measure of $\angle A$ the same as those that you obtained using the tangent and sine ratios?

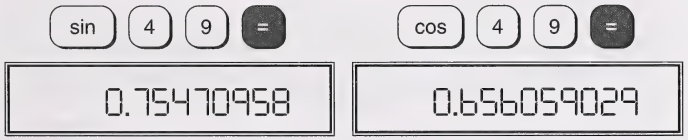
Check your answers by turning to the Appendix.

You can use a scientific calculator to find the sine and cosine ratios for given angles similarly to how you found the tangent ratio of a given angle earlier in this activity.

Example 1

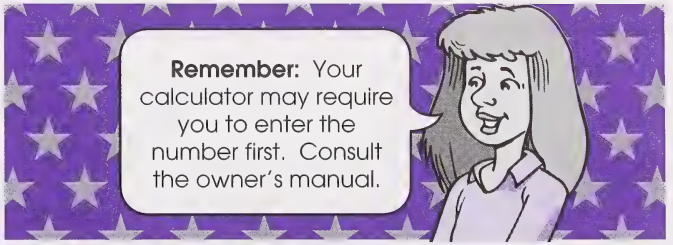
Find the sine and cosine of 49° using a scientific calculator. Round your answers to four decimal places.

Solution



$\therefore$ sin 49° = 0.7547

$\therefore$ cos 49° = 0.6561



Use a scientific calculator to answer questions 27 and 28.

27. Find the sine and cosine ratios for the following angles. Round the values to four decimal places.

a. 30° **b.** 87° **c.** 9°

d. 45° **e.** 60°

28. Study the sine and cosine ratios for the angles in question 27.

a. Predict the value of sin 0°, sin 90°, cos 0°, and cos 90°. Check your predictions with your calculator.

b. At what angle are the sine and cosine ratios the same?

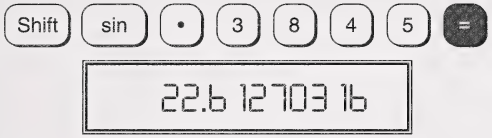
Check your answers by turning to the Appendix.

You can use a scientific calculator to find an angle given the sine or cosine ratio. Look at the following examples.

Example 2

If $\sin A = 0.3845$, then find the measure of $\angle A$. Round your answer to the nearest degree.

Solution

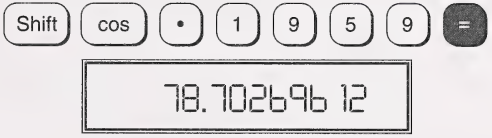


$\therefore \angle A \doteq 23°$

Example 3

If $\cos A = 0.1959$, then find the measure of $\angle A$. Round your answer to the nearest degree.

Solution

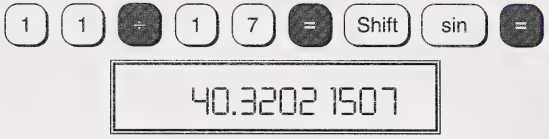


$\therefore \angle A \doteq 79°$

Example 4

If $\sin A = \frac{11}{17}$, then find the measure of $\angle A$. Round your answer to the nearest degree.

Solution



$\therefore \angle A \doteq 40°$

Use a scientific calculator to answer question 29.

29. Find the angle that is equivalent to the following trigonometric ratios. Round your answers to the nearest degree.

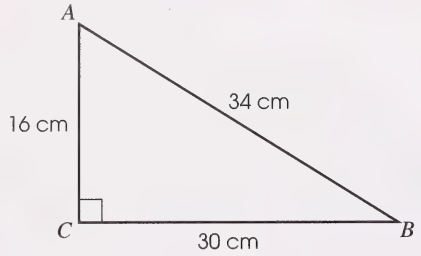
a. $\sin C = 0.82$

b. $\cos E = 0.0814$

c. $\sin A = \frac{5}{8}$

d. $\cos B = \frac{16}{23}$

30. Use the diagram given to answer the questions that follow.



a. Find the sine and cosine ratios, to four decimal places, for each acute angle.

b. What do you notice about the ratio for sin A and the ratio for cos B? What about sin B and cos A?

Check your answers by turning to the Appendix.

You solved problems in Activity 2 concerning right triangles by using the Pythagorean Theorem. The scope of the problems that you could solve was limited. Now, with the use of the primary trigonometric ratios, you can solve a number of different types of problems involving right triangles.

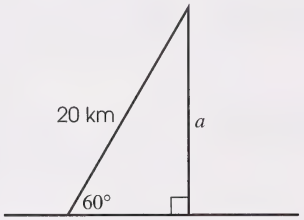
Example 5

NASA

A rocket leaves the launching pad and moves into a trajectory at an elevation of 60° from the ground (assume the angle is 60° from the launch pad). If it travels along a straight line for the first 20 km, what is its altitude at that point?

Solution

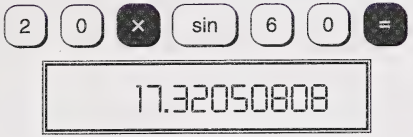
Draw a diagram.



Find a, the altitude, using a trigonometric ratio.

$$\sin 60° = \frac{a}{20}$$

$$a = 20 \times \sin 60°$$



The rocket is about 17.3 km above the ground after travelling 20 km through the air.

Now solve the following problems.

31. Given the following data about the Jasper Tramway in Jasper, Alberta, find the approximate angle of inclination (elevation) of the lift.

- Lower terminal: 1408 m above sea level
- Upper terminal: 2500 m above sea level
- Length of track: 2000 m

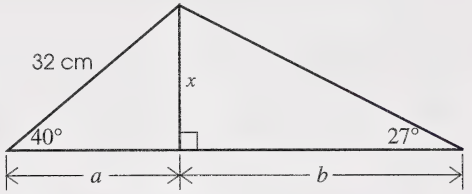
Round your answer to the nearest degree.

32. An airplane flying at a speed of 50 m/s increases its altitude at a rate of 7.5 m/s. Find the angle at which the airplane is climbing. Round your answer to the nearest degree.

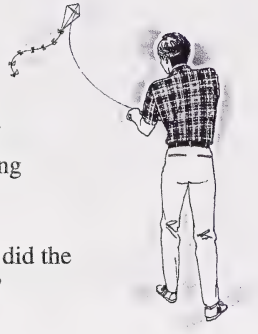
33. How far up the wall will an 8-m ladder reach if it must be slanted at 65° with the ground for safety regulations. Round your answer to two decimal places.

34. The altitude of an equilateral triangle is 19 cm. Find the length of a side to the nearest centimetre.

35. Find the measure of a and b to the nearest centimetre.



36. Karl was flying his kite on Saturday morning. He wanted to know how high his kite was flying above the ground. He attached the kite's 45-m string to the ground and then paced off 27 m to a point where the kite was flying directly over his head.



a. What angle (to the nearest degree) did the kite's string form with the ground?

b. What angle did the kite string form with a vertical line from the kite to the ground?

c. How high (to the nearest metre) is Karl's kite from the ground?

d. Describe what other methods you could have used to answer question 36.c.

Check your answers by turning to the Appendix.

Did You Know?

The word *trigonometry* is derived from three Greek words.

tri	*gono*	*metry*
↓	↓	↓
three	angle	measurement

Trigonometry has important applications in surveying, navigation, construction work, electronics, advanced chemistry, and other occupational fields.

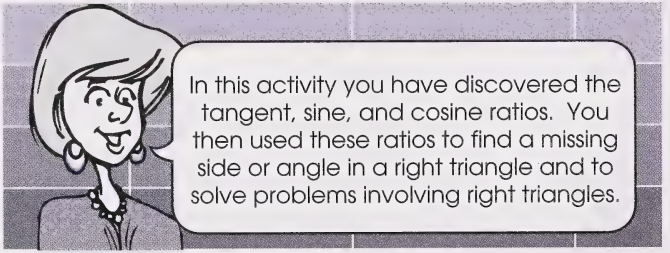
If you have access to the Internet, you may wish to search for information on the history of trigonometry or on how trigonometry is used in some of the fields mentioned previously. If you do not have access to the Internet, you may research the various topics at your local school or library.

Now Try This

Use one of the problem-solving strategies to solve the following problem.

37. Shawna walked 60 m east, then 20 m north, 20 m east, and 40 m north. How far is she from her starting point (as measured in a straight line)?

Check your answers by turning to the Appendix.

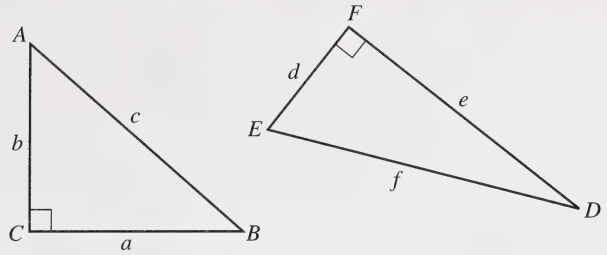


Follow-up Activities

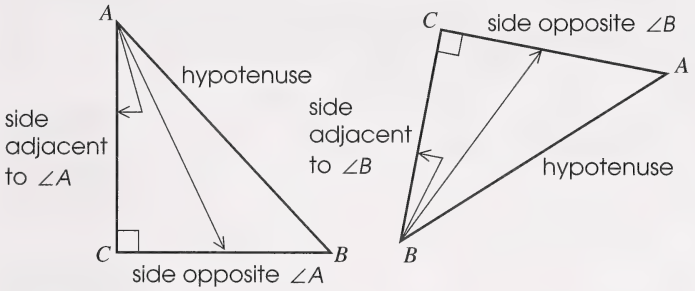
If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

It is very important that you have the ability to label right triangles correctly and are able to identify the opposite side and adjacent side in reference to both acute angles. Review the diagrams that follow.



The pattern is followed regardless of what letters are used to name the vertices.



To help you recall the three primary trigonometric ratios, you might use an acronym like the following:

$$\sin\theta = \frac{\textbf{o}\text{pposite}}{\textbf{h}\text{ypotenuse}} \qquad \cos\theta = \frac{\textbf{a}\text{djacent}}{\textbf{h}\text{ypotenuse}} \qquad \tan\theta = \frac{\textbf{o}\text{pposite}}{\textbf{a}\text{djacent}}$$

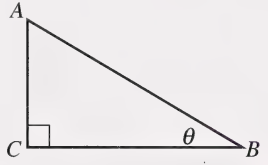
SOH	*CAH*	*TOA*
Soak	A	Toe

This phrase may help you remember the letter arrangement, or you may be able to come up with a better one yourself.

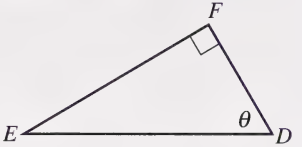
Use whatever helps you to remember these ratios.

1. Identify the adjacent side, the opposite side, and the hypotenuse in reference to $\angle\theta$ in each of the following triangles.

a.

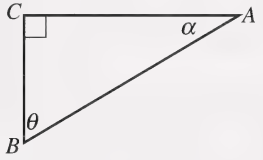


b.

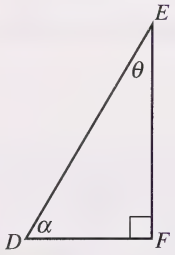


2. Use small letters to name the primary trigonometric ratios for $\angle\theta$ and $\angle\alpha$ in each of the following triangles.

a.



b.

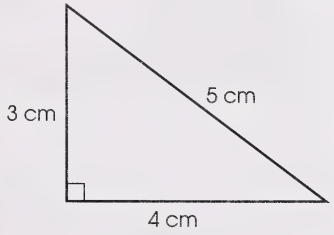


Check your answers by turning to the Appendix.

Enrichment

In previous mathematics courses you discovered the Pythagorean Theorem, which relates the areas of squares on the sides of right triangles. Might this property of right triangles be true for figures other than squares? Check it out as you do the following questions.

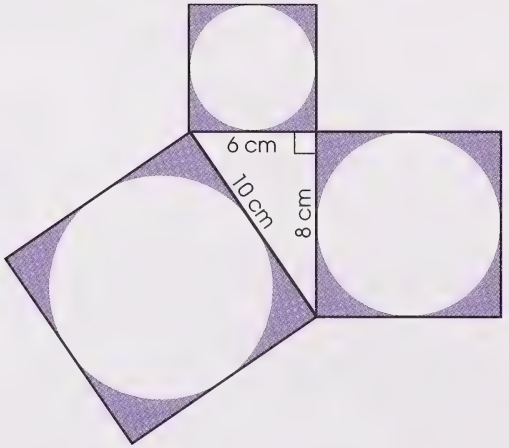
1. a. Copy the following right triangle and draw isosceles right triangles on each side.



b. Measure the base and height and find the area of each triangle. Round each area to one decimal place.

c. Is there any relationship? Explain.

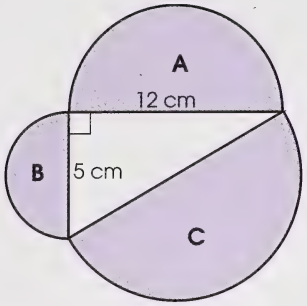
2. a. Find the areas of the circles on each side of the following right triangles. Round each area to two decimal places.



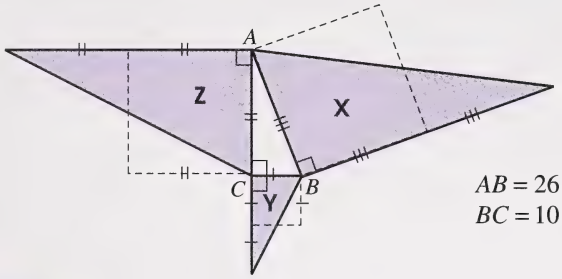
b. Is there any relationship? Explain.

3. What conclusion can you draw? Explain.

4. Find the area of each semicircle without doing any measuring. Round each area to two decimal places.



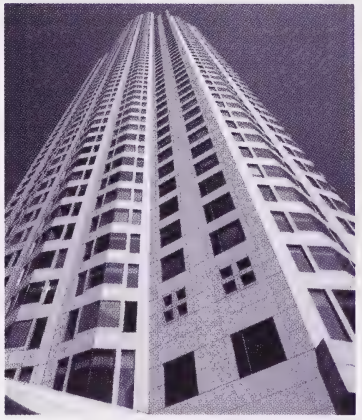
5. Find the area of each triangle on the sides of ΔABC without doing any measuring.



$AB = 26$ cm
$BC = 10$ cm

Check your answers by turning to the Appendix.

Conclusion



In this section you used the concepts involved in congruent and similar triangles to develop the primary trigonometric ratios. In turn, you used the primary trigonometric ratios to solve problems involving right triangles. You constructed tools to help you apply your knowledge of trigonometry to solve practical problems involving measurement and right triangles.

Now you can probably find the height of any tower or tall building using the concepts studied in this module. Would you be able to find the height of the building in the photograph using a clinometer?

Assignment

Assignment Booklet

You are now ready to complete the assignment for Section 3.

Module Summary

In Module 5 you analysed the motions of figures (such as triangles) on a coordinate plane and discovered the rules that could be used to represent these motions. You drew two-dimensional views of three-dimensional figures, and you drew and built three-dimensional models from two-dimensional plans. You developed the primary trigonometric ratios and solved right triangles. You also discovered how to make indirect measurements using trigonometric ratios and some simple homemade measuring instruments.

You can now determine the shortest route to your friend's house and are better able to understand the geometry involved in building design.

Do you now pay more attention to how and where measurement is used in everyday activities and in the structures you see?

Final Module Assignment

You are now ready to complete the final module assignment.

APPENDIX

Glossary

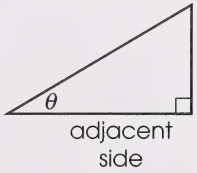
Suggested Answers

Formula Sheet

Dot Paper, Grids, and Cutouts

Glossary

Adjacent side: the side next to the reference angle that is not the hypotenuse



Angle of depression: the angle formed between the horizon line and the line of sight to the object, which is below the observer's eye

Angle of elevation: the angle formed between the horizontal plane and the line of sight to the object, which is above the observer's eye

Angle of rotation: the number of degrees (usually 90°, 180°, or 270°) through which a figure is turned either clockwise or counterclockwise about a point called the turn centre

Congruent figures: figures that have the same size and shape

Cosine: the ratio formed by dividing the length of the side adjacent to the reference angle by the length of the hypotenuse

Dilatation: a transformation that enlarges or reduces a figure without changing its shape

Hypotenuse: the side opposite the right angle in a right triangle; the longest side

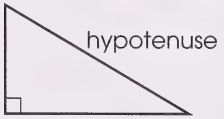


Image: the resultant figure after performing a transformation on a figure

Line of symmetry: a line that divides a figure into congruent parts that are reflection images of each other

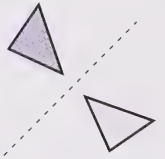
Locus of points: the set of all points that satisfy a given condition

Mapping rule: directions for moving a figure to find its image

$(x, y) \rightarrow (x+1, y+3)$

Indicates that the new x-coordinate is increased by 1 and the y-coordinate by 3.

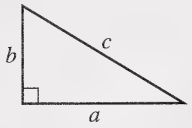
Mirror line: a line about which a figure is reflected or flipped



Opposite side: the side opposite the reference angle

Pythagorean Theorem: a theorem in which the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides

$c^2 = a^2 + b^2$



Reflection: a transformation that is represented by a flipping motion about a mirror line

Reflection line: another name for a mirror line

Rotation: a transformation that is represented by a turning motion about a stationary point

Rotational symmetry: the ability of a figure to map onto itself after a partial turn about its centre

Scale factor: the ratio of the lengths of the sides of the image to the lengths of the sides of the original figure

$(x, y) \rightarrow (ax, ay)$

the scale factor

Similar figures: figures that have the same shape and congruent corresponding angles

Sine: the ratio formed by dividing the length of the side opposite the reference angle by the length of the hypotenuse

Tangent: the ratio formed by dividing the length of the side opposite the reference angle by the length of the side adjacent to the reference angle

Transformation: the changing of a figure's shape, size, or position

Translation: a transformation that is represented by a sliding motion with no turns or other motions

Translation vector: an arrow that denotes distance and direction that an object is to be translated

$(4, 2)$ ← 4 to the right and 2 up

Trigonometry: the study of the relationships of angles and sides in right triangles

Turn centre: the point about which a figure is rotated

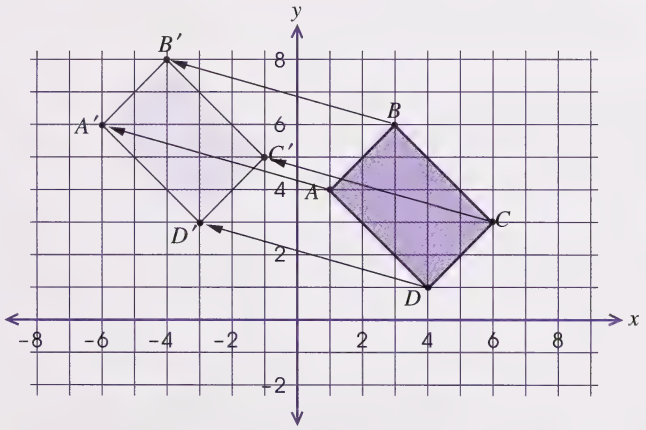
Suggested Answers

Section 1: Activity 1

1. The measures of the length of the sides and the size of the angles are the same in both rectangles.

2. Each point of rectangle *ABCD* has been moved 7 units to the left and 2 units up.

3. Your arrows should look similar to the following:



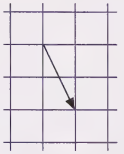
The arrows are the same length and are parallel.

4. The corresponding sides of the original rectangle and those of its image are parallel.

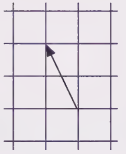
5. Yes, rectangle $A'B'C'D'$ would fit exactly onto rectangle *ABCD*.

6. Yes, rectangle *ABCD* and rectangle $A'B'C'D'$ are congruent.

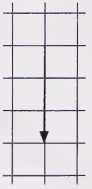
7. a.

Words	Ordered Pair	Arrow
The image is translated 1 unit to the right and 2 units down.	$(1,-2)$	

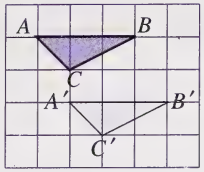
b.

Words	Ordered Pair	Arrow
The image is translated 1 unit to the left and 2 units up.	$(-1,2)$	

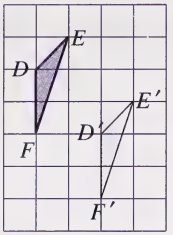
c.

Words	Ordered Pair	Arrow
The image is translated 3 units down.	$(0,-3)$	

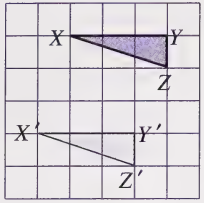
8. **a.**



b.



c.



9. Figures B, D, and F are translation images. The ordered pairs for the translations are $(5,-3)$, $(-2,3)$, and $(2,2)$ respectively.

10. **a.** $T'(1,5)$, $O'(-1,1)$, and $P'(1,-1)$

b. The corresponding sides are the same length.

c. The measures of the corresponding angles are the same.

d. Yes, the two triangles are congruent because they are the same size and shape. The corresponding sides and angles are equal.

e. If you add 4 to each x-coordinate of the vertices of ΔTOP, you will get the x-coordinate of the vertices of $\Delta T'O'P'$.

f. If you add 2 to each y-coordinate in ΔTOP, you will get the y-coordinate in $\Delta T'O'P'$.

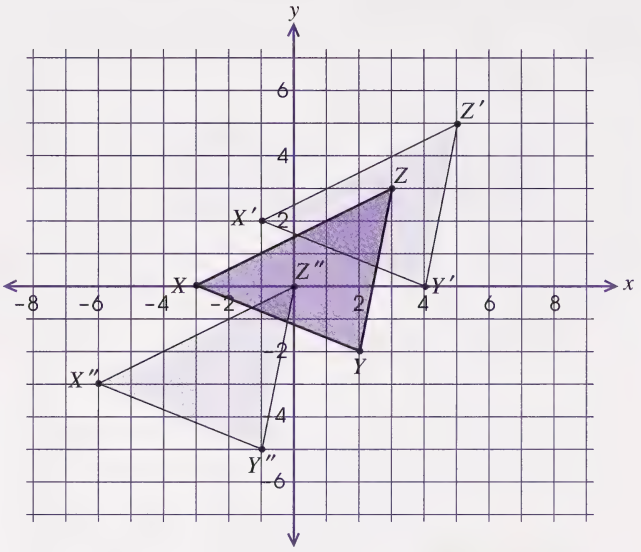
11. **a.** $P'(-7+5, 2+3) = P'(-2,5)$

b. $W'(6+5, 4+3) = W'(11,7)$

c. $M'(-2+5, -5+3) = M'(3,-2)$

d. $N'(0+5, 9+3) = N'(5,12)$

12. **a.**



b. Refer to $\Delta X''Y''Z''$ in the answer to question 12.a.

13. Yes, the sides of $\Delta X'Y'Z'$ are parallel to the sides of $\Delta X''Y''Z''$.

14. 5 units to the left and 5 units down

15. The translation is 4 units to the right and 2 units down.

16. **a.** $A(3,1)$, $B(7,3)$, $C(6,-3)$

b. $A(-4,1)$, $B(-1,3)$, $C(3,-1)$

17. **a.** Answers will vary. A translation of less than 2 units to the left and more than 7 units down is required. Some examples are $(-1,-8)$, $(0,-8)$, and $(0,-9)$.

b. Answers will vary. A translation of more than 7 units to the left and more than 7 units down is required. Some examples are $(-8,-8)$, $(-8,-10)$, and $(-10,-9)$.

18. **a.** Yes, the plane's path is heading towards the coordinates $(0,0)$, where the airport is located.

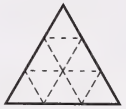
b. The airplane should touchdown at 11:55 $(5+5+10=20$ min after the plane was first spotted).

Now Try This

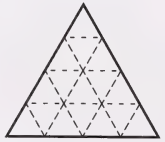
19.

There are 25 of these triangles in the diagram.

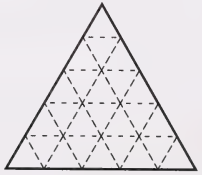
There are 13 of these triangles in the diagram.



There are 6 of these triangles in the diagram.



There are 3 of these triangles in the diagram.



There is only 1 of these triangles in the diagram.

Thus, there are $25+13+6+3+1=48$ triangles in the diagram.

Section 1: Activity 2

1. Yes, the lengths of the corresponding sides are the same.

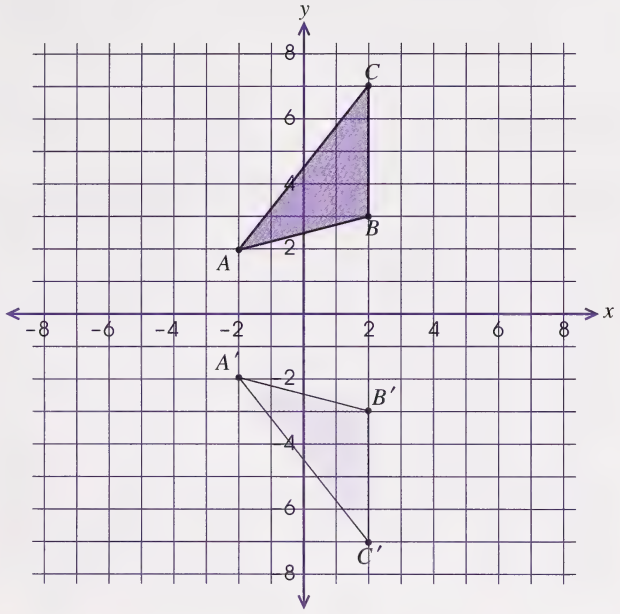
2. Yes, the measures of the corresponding angles are the same.

3. No, quadrilateral $A'B'C'D'$ will not cover quadrilateral $ABCD$ completely.

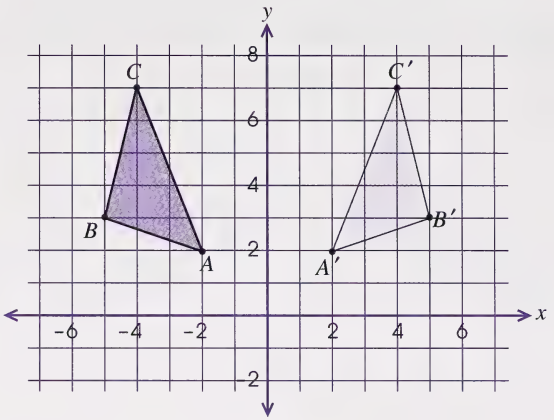
4. Quadrilateral $A'B'C'D'$ is not a translation of quadrilateral $ABCD$ because you cannot slide quadrilateral $A'B'C'D'$ onto quadrilateral $ABCD$ and have it cover completely.

5. You can fit the quadrilateral $A'B'C'D'$ onto quadrilateral $ABCD$ by flipping it over and sliding it onto quadrilateral $ABCD$.

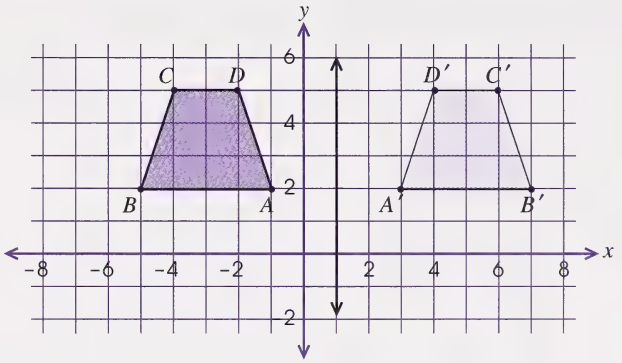
6. a.



b.



c.



7. a. Each point and its reflection are the same perpendicular distance from the line of reflection.

b. A figure and its reflection are congruent.

8. Figures A and D are not reflection images of figure 1. Figure A is not congruent to figure 1, and figure D is a translation of figure 1.

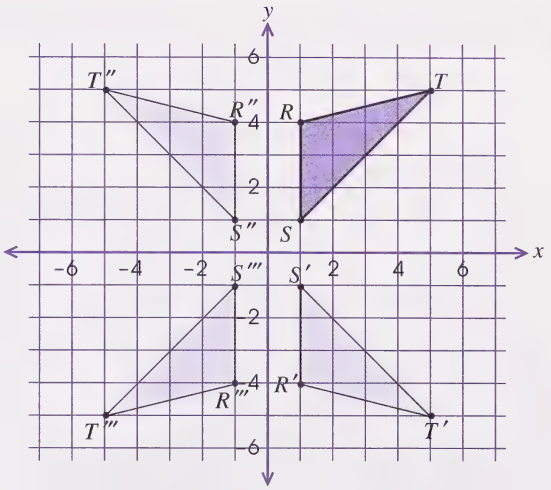
9.

Point	Reflection Line	
	x-axis	*y*-axis
$(2, 3)$	$(2, -3)$	$(-2, 3)$
$(-3, 4)$	$(-3, -4)$	$(3, 4)$
$(-1, -5)$	$(-1, 5)$	$(1, -5)$
$(0, 3)$	$(0, -3)$	$(0, 3)$
$(4, 0)$	$(4, 0)$	$(-4, 0)$
$(4, -1)$	$(4, 1)$	$(-4, -1)$

10. **a.** $(x, y) \rightarrow (\ x\ ,\ -y\ )$ **b.** $(x, y) \rightarrow (\ -x,\ y\ )$

11. In a reflection about the *x*-axis, each point has the same *x*-coordinate and the opposite *y*-coordinate. In a reflection about the *y*-axis, each point has the same *y*-coordinate and the opposite *x*-coordinate.

12. **a.**



b. Yes, because all the properties of a reflection are met.

c. Yes, because all the properties of a reflection are met.

d. No, this is not a reflection image of ΔRST because a flip about a line between ΔRST and $\Delta R'''S'''T'''$ will not map ΔRST exactly onto $\Delta R'''S'''T'''$.

e. $\Delta R'''S'''T'''$ is a reflection of $\Delta R'S'T'$ about the *y*-axis and a reflection of $\Delta R''S''T''$ about the *x*-axis.

13. **a.** $M'(-2, -3)$, $N'(2, -4)$, and $O'(1, 1)$

b. $M''(2, 3)$, $N''(-2, 4)$, and $O''(-1, -1)$

14. **a.** The x-coordinate does not change.

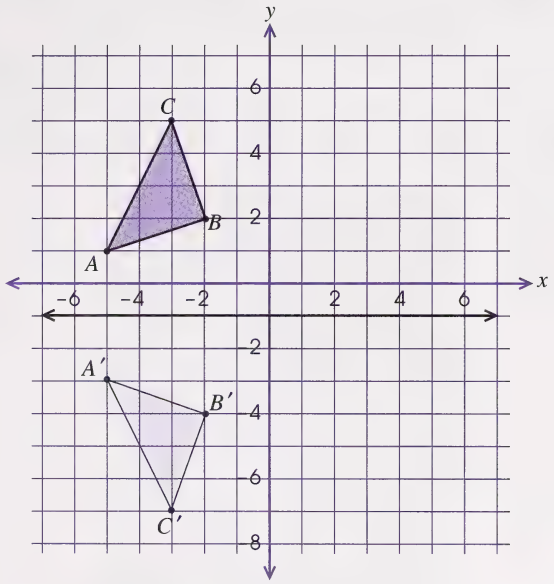
b. The y-coordinate does not change.

c. The point must be located on the x-axis.

d. The point must be located on the y-axis.

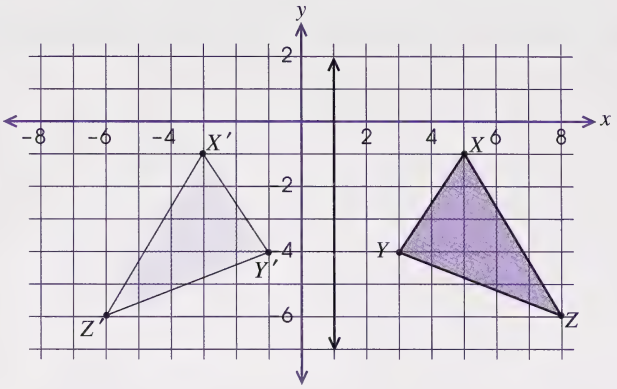
e. The point $(0, 0)$ does not change after a reflection about either axis.

15.



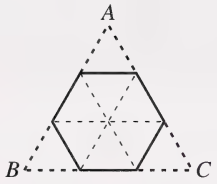
The equation of the line of reflection is $y = -1$.

16.



The equation of a line of reflection is $x = 1$. Other vertical lines left of $x = 1$ are acceptable as well.

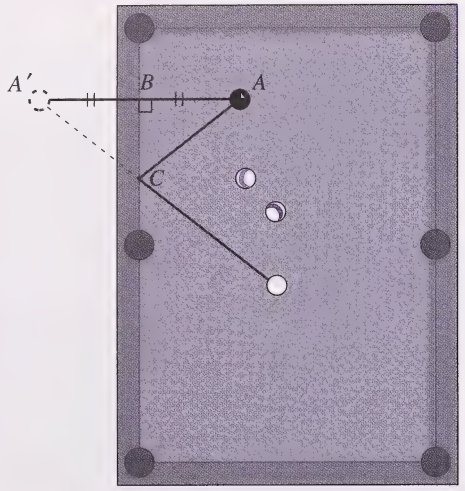
17. **a.**



The shape formed is a regular hexagon.

b. Each transformation used is a reflection about a line.

18. Your diagram should be similar to the following:

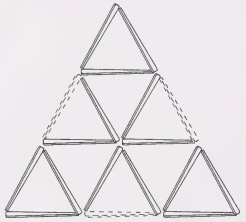


A' is the reflection image of A. In order to hit the 8-ball, Kyle has to shoot the cue ball towards point A'.

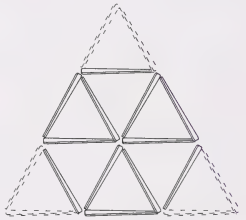
Now Try This

19. $(x, y) \rightarrow (y, x)$

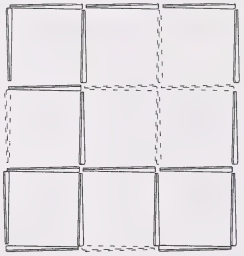
20. **a.**



b.



21. Answers will vary. A sample answer is provided.



1 large square and
3 small squares

Section 1: Activity 3

1. $\frac{1}{4}$ **turn cw**

$\Delta ABC \rightarrow \Delta A'B'C'$

$A(2,5) \rightarrow A'(\ 5\ , -2)$
$B(1,1) \rightarrow B'(1, -1\ )$
$C(4,2) \rightarrow C'(\ 2\ , -4\ )$

$\frac{1}{2}$ **turn cw**

$\Delta ABC \rightarrow \Delta A''B''C''$

$A(2,5) \rightarrow A''(-2, -5\ )$
$B(1,1) \rightarrow B''(\ -1\ , -1\ )$
$C(4,2) \rightarrow C''(\ -4\ , -2)$

$\frac{3}{4}$ **turn cw**

$\Delta ABC \rightarrow \Delta A'''B'''C'''$

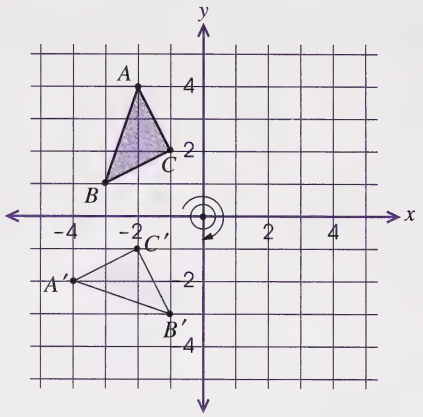
$A(2,5) \rightarrow A'''(-5\ ,\ 2\ )$
$B(1,1) \rightarrow B'''(-1,\ 1\ )$
$C(4,2) \rightarrow C'''(\ -2\ ,4)$

2. a. The image and the original figure are congruent.

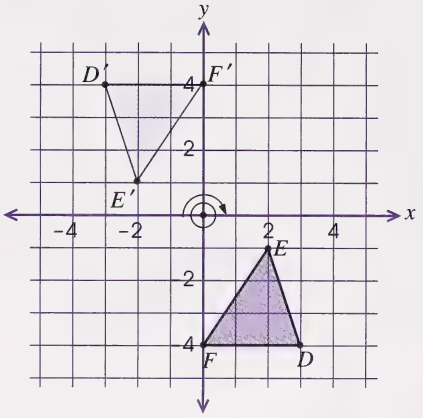
b. The orientation of the figure changes.

c. A point and its image are the same distance from the turn centre.

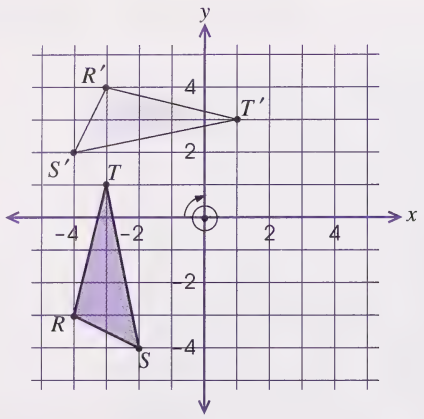
3. a.



b.



c.



4. $\Delta ABC \to \Delta A'B'C'$

$A(-2,4) \to A'(-4,-2)$
$B(-3,1) \to B'(-1,-3)$
$C(-1,2) \to C'(-2,-1)$

$\Delta DEF \to \Delta D'E'F'$

$D(3,-4) \to D'(-3,4)$
$E(2,-1) \to E'(-2,1)$
$F(0,-4) \to F'(0,4)$

$\Delta RST \to \Delta R'S'T'$

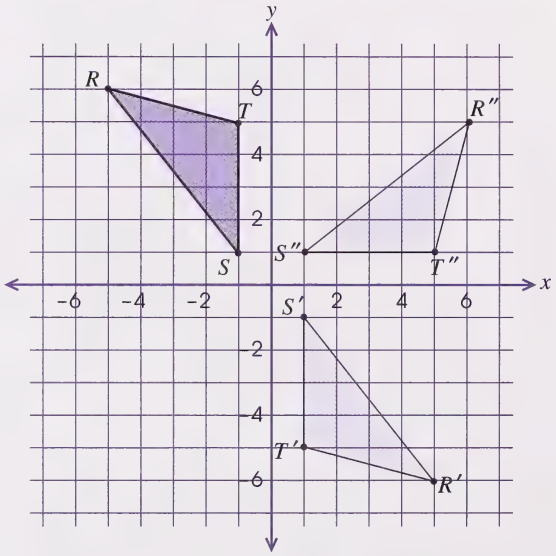
$R(-4,-3) \to R'(-3,4)$
$S(-2,-4) \to S'(-4,2)$
$T(-3,1) \to T'(1,3)$

5. $\frac{1}{4}$ turn cw: $(x,y) \to (\ y\ , -x\ )$

$\frac{1}{2}$ turn cw: $(x,y) \to (-x\ , -y\ )$

$\frac{3}{4}$ turn cw: $(x,y) \to (-y\ ,\ x\ )$

6.



a. Yes, $\Delta R'S'T'$ is a rotation image of ΔRST. If you rotate ΔRST a half turn about $(0,0)$, it fits exactly on $\Delta R'S'T'$.

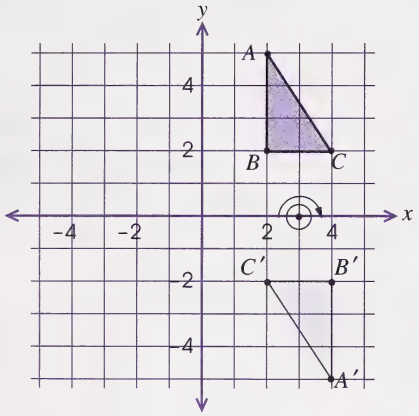
b. Yes, $\Delta R''S''T''$ is a rotation image of ΔRST. If you rotate ΔRST a quarter turn clockwise (or a three-quarter turn counterclockwise) about $(0,0)$, it fits exactly on $\Delta R''S''T''$.

7. a. $M'(3,2)$, $N'(-1,-1)$, $O'(4,-2)$

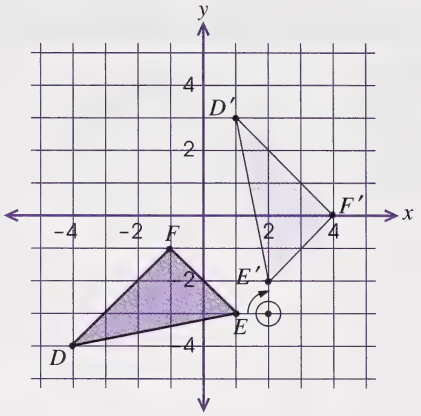
b. $M''(2,-3)$, $N''(-1,1)$, $O''(-2,-4)$

c. $M'''(-3,-2)$, $N'''(1,1)$, $O'''(-4,2)$

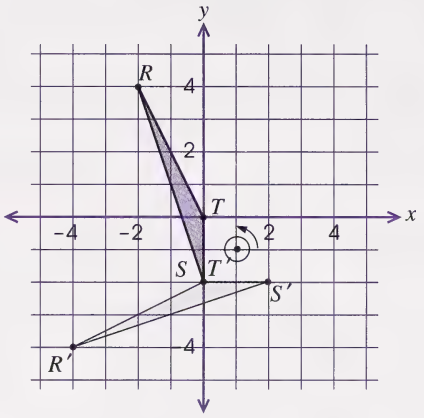
8. a.



b.



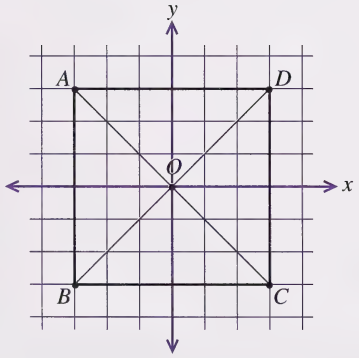
c.



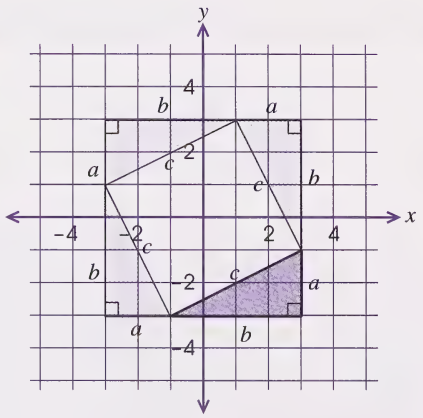
9.

Point	Rotation		
	90° cw	180° cw	270° cw
$(0, 4)$	$(4, 0)$	$(0, -4)$	$(-4, 0)$
$(-3, 0)$	$(0, 3)$	$(3, 0)$	$(0, -3)$
$(-2, -3)$	$(-3, 2)$	$(2, 3)$	$(3, -2)$
$(3, 5)$	$(5, -3)$	$(-3, -5)$	$(-5, 3)$
$(1, -4)$	$(-4, -1)$	$(-1, 4)$	$(4, 1)$

10. Draw a square centred on a coordinate grid. Label it *ABCD*. Draw in the diagonals and label their intersection point *O*. Rotate ΔAOD through a $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ turn about *O*. It maps exactly onto ΔDOC, ΔCOB, and ΔBOA. Therefore, $\Delta AOD \cong \Delta DOC \cong \Delta COB \cong \Delta BOA$.



11. a.



b. The shape of the outside figure is a square.

c. The shape of the inside figure is a square.

d. $6 \times 6 = 36$

The area of the outside square is 36 square units.

e. The length of a side of the inside square is equal to the length of the hypotenuse of the original triangle.

$$c^2 = a^2 + b^2$$
$$c^2 = 4^2 + 2^2$$
$$c^2 = 16 + 4$$
$$c^2 = 20$$
$$c = \sqrt{20}$$

Now find the area of the inside square.

$$\begin{aligned} A &= \sqrt{20} \times \sqrt{20} \\ &= \sqrt{20 \times 20} \\ &= \sqrt{400} \\ &= 20 \end{aligned}$$

The area of the inside square is 20 square units.

f. Area of each triangle $= \frac{1}{2}bh$

$$= \frac{1}{2}(4)(2)$$
$$= 4$$

Area of outside square = Area of inside square
$+ 4 \times$ Area of each triangle

$$36 = 20 + 4(4)$$
$$36 = 20 + 16$$
$$36 = 36$$

12. a. Answers will vary. Some examples are provided.

Via Rail

Bank of Nova Scotia

b. Answers will vary. A sample answer is given.

Now Try This

13. a. The word AIM can be reflected about a vertical line.

A
I
M

b. The word OXO can be transformed in all three ways and still read the same.

c. The word NOON can be transformed by rotating it a $\frac{1}{2}$ turn.

d. The word DEED can be transformed by reflecting it about a horizontal line.

e. The word HOAX can be transformed by reflecting it about a vertical line.

H
O
A
X

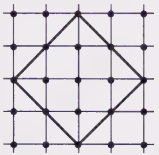
14. Answers will vary. Some sample answers have been provided.

- reflected about a vertical line
  TAM TOOTH IOWA HOW
- reflected about a horizontal line
  HOBO BOX DECODE
- rotated a half turn
  SWIMS MOW

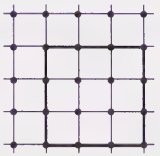
You have to visualize M changed to a W.

15. a.

b.

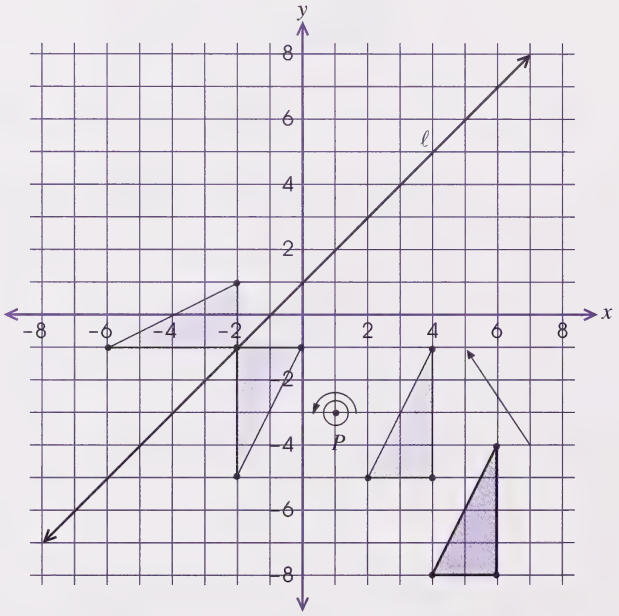


c.

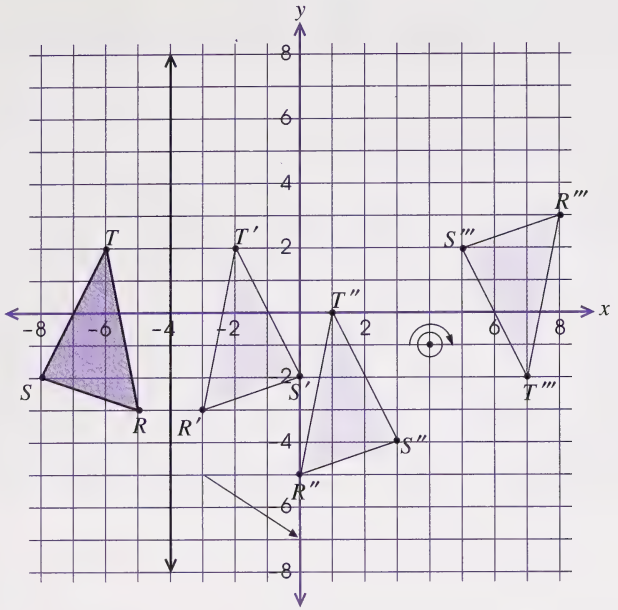


Section 1: Activity 4

1.



2.



The first motion is a reflection about the line $x=-4$. Next, a translation of 3 units to the right and 2 units down is performed. The last motion is a rotation of 180° cw about point $(4,-1)$.

3. **a.** $A(x,y)\rightarrow B(x+4,y)$
$B(x,y)\rightarrow C(x,-y)$

b. $A(x,y)\rightarrow B(-x,-y)$
$B(x,y)\rightarrow C(-x,y)$

4. $A(x,y)\rightarrow C(x,-y)$

5. In question 3.a., the position of figure *C* would not change. Figure *A* would reflect into the third quadrant and then slide to figure *C* in the fourth quadrant.

In question 3.b., the position of figure *C* would not change. Figure *A* would reflect into the fourth quadrant and then rotate to figure *C* in the second quadrant.

6. **a.** a reflection about the *y*-axis or $(x,y)\rightarrow(-x,y)$

b. a reflection about the line $y=x$ or $(x,y)\rightarrow(y,x)$

c. a $\frac{1}{2}$ turn about $(0,0)$ or $(x,y)\rightarrow(-x,-y)$

d. a $\frac{1}{4}$ turn clockwise about $(0,0)$ or $(x,y)\rightarrow(y,-x)$

7. **a.** a $\frac{1}{2}$ turn about $(0,0)$

b. a translation 2 units to the left and 3 units up

c. a reflection about the *y*-axis

d. a $\frac{3}{4}$ turn clockwise about $(0,0)$

e. a reflection about the *x*-axis

f. a $\frac{1}{4}$ turn clockwise about $(0,0)$

8.

Coordinates of the Original			Coordinates of the Image			Type of Transformation
A	B	C	A'	B'	C'	
$(2, 5)$	$(3, 4)$	$(6, 2)$	$(3, 3)$	$(4, 2)$	$(7, 0)$	translation
$(-3, 2)$	$(-5, 6)$	$(-1, 4)$	$(3, -2)$	$(5, -6)$	$(1, -4)$	$\frac{1}{2}$ turn about $(0, 0)$
$(1, 0)$	$(-2, 5)$	$(-4, -3)$	$(-1, 0)$	$(2, 5)$	$(4, -3)$	reflection about the y-axis
$(4, 3)$	$(5, -2)$	$(2, 1)$	$(-3, 4)$	$(2, 5)$	$(-1, 2)$	$\frac{3}{4}$ turn cw about $(0, 0)$
$(-3, -5)$	$(-2, 1)$	$(-4, 3)$	$(-3, 5)$	$(-2, -1)$	$(-4, -3)$	reflection about the x-axis

Now Try This

9. a. This design uses reflections and translations to create the series of figures.

b. This design uses rotations and translations to create the series of figures.

Section 1: Activity 5

1. a. $AB = 2.5$ cm $\quad A'B' = 5.0$ cm
$AC = 2.5$ cm $\quad A'C' = 5.0$ cm
$BC = 1.8$ cm $\quad B'C' = 3.6$ cm

The lengths of the sides of $\Delta A'B'C'$ are twice those of ΔABC.

b. $\angle A = 42°$ $\quad \angle A' = 42°$
$\angle B = 68°$ $\quad \angle B' = 68°$
$\angle C = 70°$ $\quad \angle C' = 70°$

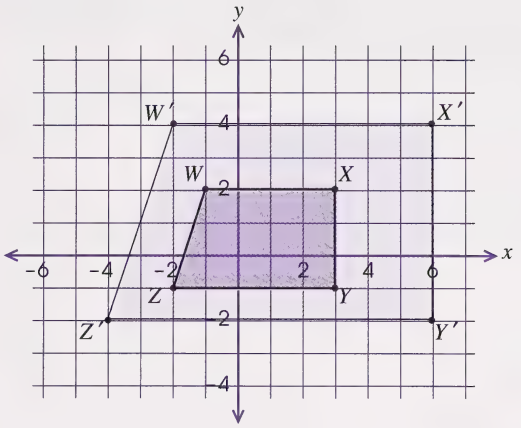
The corresponding angles of ΔABC and $\Delta A'B'C'$ have the same measure.

c. The corresponding sides are parallel.

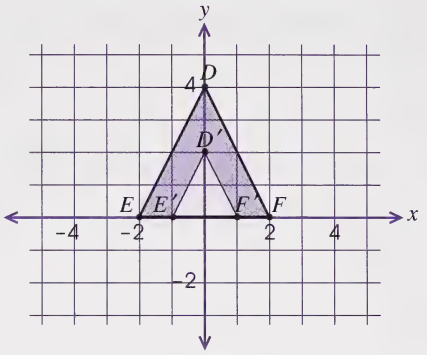
2. The scale factor is 2.

3. $(x, y) \rightarrow (2x, 2y)$

4. The vertices of the image of *WXYZ* are $W'(-2, 4)$, $X'(6, 4)$, $Y'(6, -2)$, and $Z'(-4, -2)$.

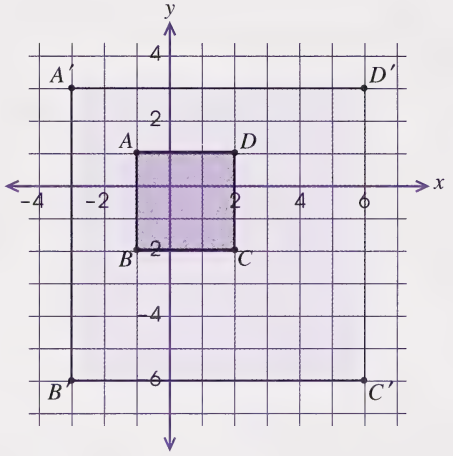


5. a.



b. $(x, y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$

6.



7. **a.** Area of $\Delta ABC = \frac{1}{2}(4)(3)$

$= 6$ square units

Area of $\Delta A'B'C' = \frac{1}{2}(8)(6)$

$= 24$ square units

b. The area of $\Delta A'B'C'$ is equal to the scale factor squared times the area of ΔABC. That is, $24 = 2^2 \times 6$

$24 = 4 \times 6$

8. **a.** The coordinates of the vertices and the lengths of the corresponding sides change.

b. The measure of the corresponding angles, the orientation of the figure, and the slope of the sides remain the same.

9.

Original	Image	Mapping Rule
$(3, -1)$	$(6, -2)$	$(x, y) \rightarrow (2x, 2y)$
$(4, 2)$	$(2, 1)$	$(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$
$(-4, 6)$	$(-2, 3)$	$(x, y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$
$(4, 6)$	$(6, 9)$	$(x, y) \rightarrow (1.5x, 1.5y)$
$(3, 2)$	$(9, 6)$	$(x, y) \rightarrow (3x, 3y)$

10. **a.** The scale factor is $\frac{1}{2}$.

b. The scale factor is 4.

11.

Area of Original	Area of Image	Scale Factor
24 cm^2	6 cm^2	$\frac{1}{2}$
16 cm^2	36 cm^2	1.5
36 cm^2	4 cm^2	$\frac{1}{3}$
8 cm^2	50 cm^2	2.5

Now Try This

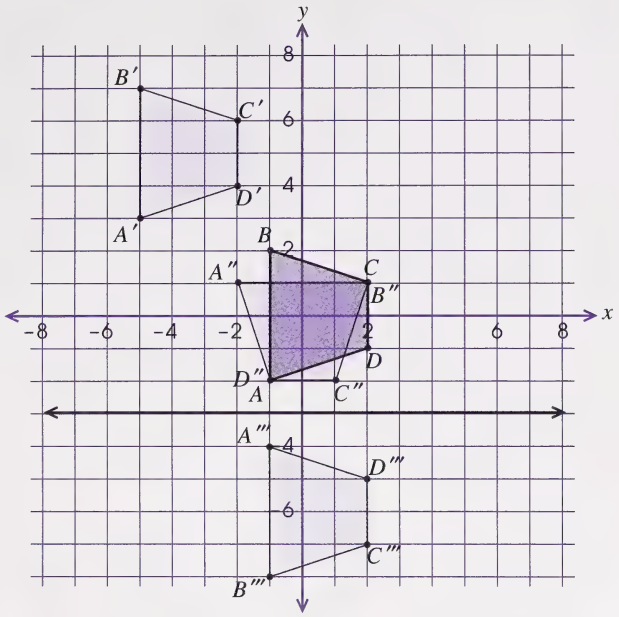
12. Answers will vary. Some examples of real-world dilatations are as follows:

- a model airplane, which is a reduction of an actual airplane
- a map of Canada, which is a reduction of the actual country
- a television screen, which is a reduction of the actual set
- a statue of a person, which may be an enlargement or a reduction of the actual person

Section 1: Follow-up Activities

Extra Help

1. a. Refer to the quadrilateral labelled *ABCD*.



b. Refer to the quadrilateral labelled $A'B'C'D'$ in the answer to question 1.a.

c. Refer to the quadrilateral labelled $A''B''C''D''$ in the answer to question 1.a. This rotation is equivalent to a rotation of 90° cw about $(0, 0)$.

d. **Question 1.b.**

$(x, y) \rightarrow (x-4, y+5)$

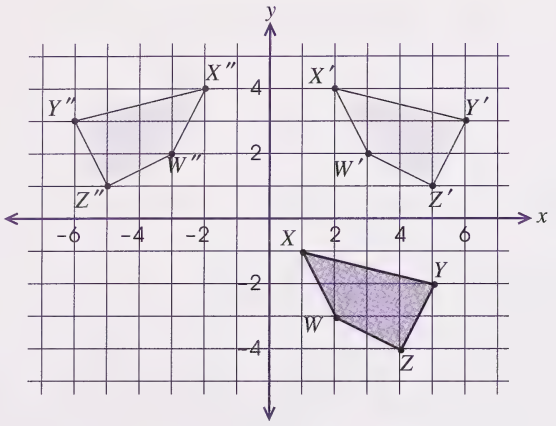
Question 1.c.

$(x, y) \rightarrow (y, -x)$

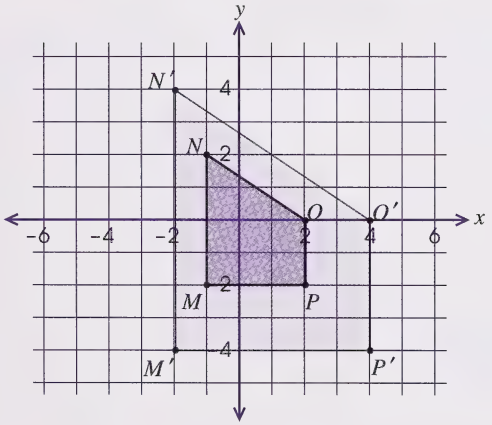
e. Refer to the quadrilateral labelled $A'''B'''C'''D'''$ in the answer to question 1.a.

2. The transformation could be a translation of at least 7 units down, a reflection about the *x*-axis, or a rotation of 270° cw (or 90° ccw) about the origin. Other combinations of translations, reflections, or rotations may be possible.

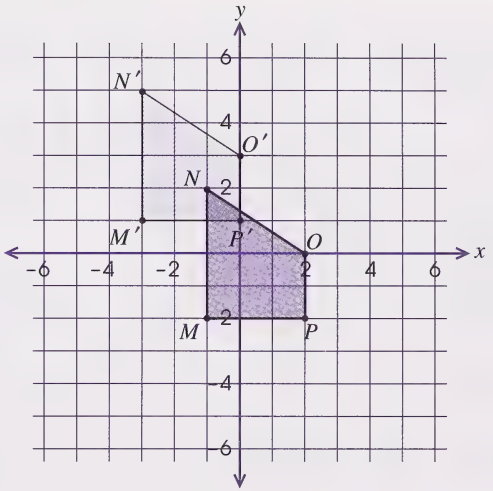
3. The following diagram shows quadrilateral $W''X''Y''Z''$ in position prior to each translation. To find each original position, you need to perform each translation in reverse. First, quadrilateral $W''X''Y''Z''$ is reflected in the *y*-axis to give quadrilateral $W'X'Y'Z'$. Then, quadrilateral $W'X'Y'Z'$ is translated 1 unit to the left and 5 units down.



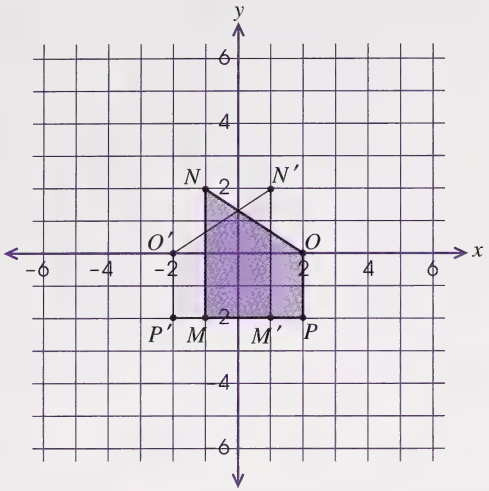
4. a. The coordinates of the new quadrilateral are $M'(-2,-4)$, $N'(-2,4)$, $O'(4,0)$, and $P'(4,-4)$.



b. The coordinates of the new quadrilateral are $M'(-3,1)$, $N'(-3,5)$, $O'(0,3)$, and $P'(0,1)$.



c. The coordinates of the new quadrilateral are $M'(1,-2)$, $N'(1,2)$, $O'(-2,0)$, and $P'(-2,-2)$.

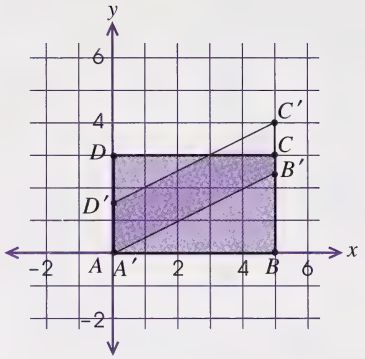


5.

Vertical line of reflection only—A, T, U, V, W, Y
Horizontal line of reflection only—B, C, D, E, K
Both lines of reflection—H, I, O, X

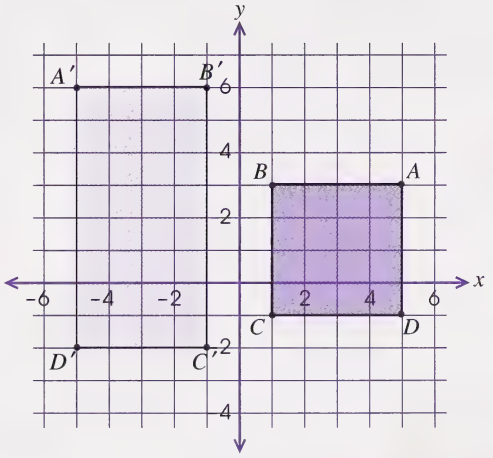
Enrichment

1. a.



b. Rectangle *ABCD* changes into a parallelogram that is squeezed vertically.

2. a.



b. Square *ABCD* is reflected about the *y*-axis and stretched vertically to become a rectangle.

c. It would be reflected about the *x*-axis and stretched horizontally to become a rectangle.

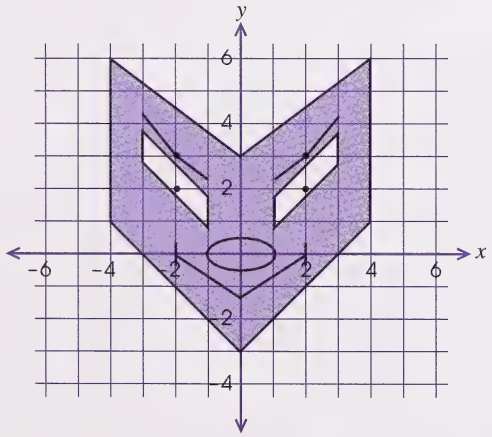
3. a. Figure *ABCD* would stay a parallelogram that is stretched horizontally and squeezed vertically.

b. Figure *ABCD* would be reflected about the *y*-axis and squeezed vertically.

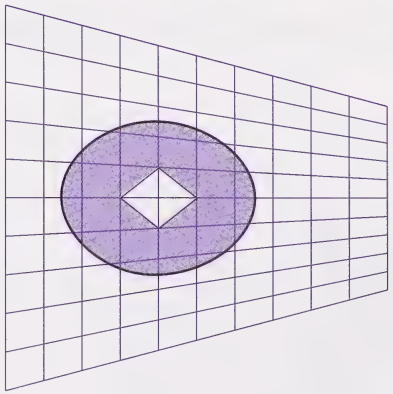
c. Figure *ABCD* would be stretched horizontally and shifted to the right.

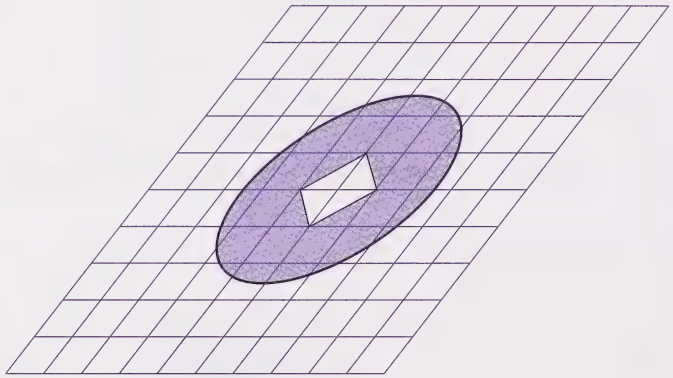
d. Figure *ABCD* would be rotated $\frac{1}{2}$ turn about $(0, 0)$, enlarged, and shifted down from the *x*-axis.

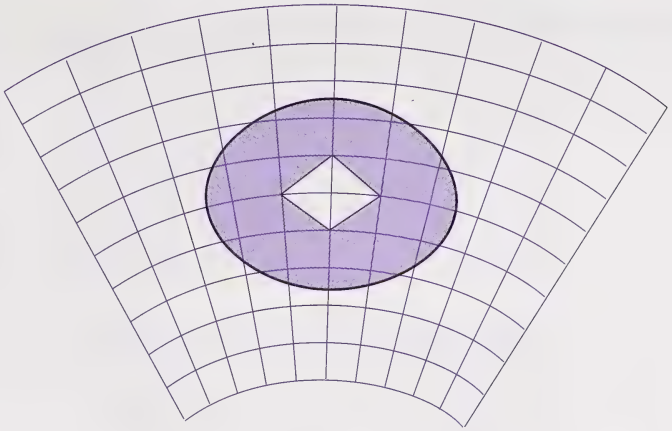
4.



5.



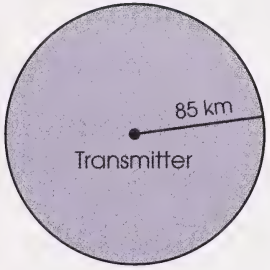




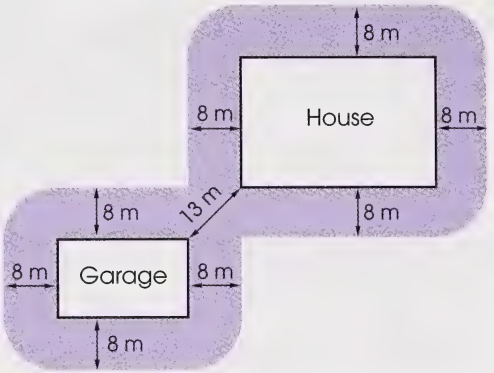
6. Yes, they are similar.

Section 2: Activity 1

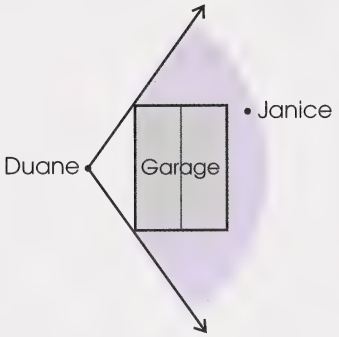
1. The locus of the area covered by the transmitter is a circle with a radius of 85 km.



2. The tree cannot be planted in the shaded area.

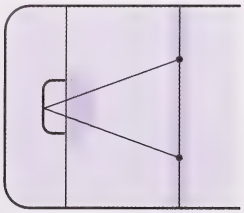


3. **a.** Janice would not be able to see Duane from any point within the shaded area between the two lines connecting him and the corners of the garage.



b. The angle of Duane's obstructed view is approximately 110°.

4. a. Sketches may vary. A sample sketch is given.

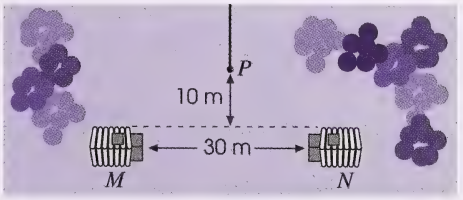


The locus is a line on the blue line.

b. The locus is a set of equally spaced points.

• • • • • •

5. The shower building could be located anywhere along the line starting at point *P*.



6. a.

Rectangles with an Area of 16 Square Units		
Length (units)	Width (units)	Perimeter (units)
1	16	34
2	8	20
4	4	16
8	2	20
16	1	34

b. The rectangle with a length of 1 unit and a width of 16 units and the rectangle with a length of 16 units and a width of 1 unit have the greatest perimeter.

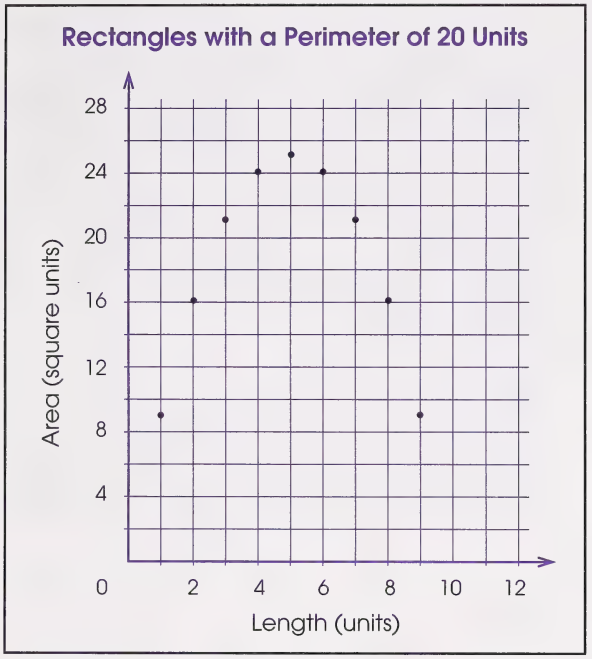
c. The rectangle with a length and width of 4 units has the least perimeter.

7. a.

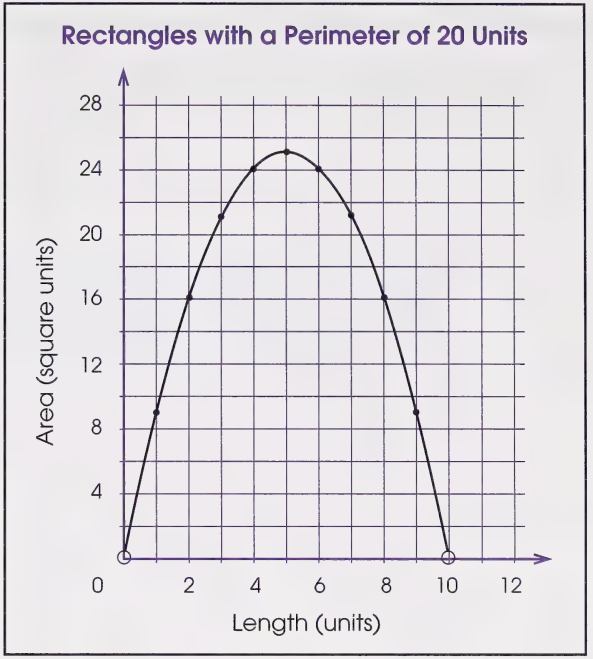
Rectangles with a Perimeter of 20 Units		
Length (units)	Width (units)	Area (square units)
1	9	9
2	8	16
3	7	21
4	6	24

5	5	25
6	4	24
7	3	21
8	2	16
9	1	9

b.



c.



The dots are joined because it is possible to have a length with a fraction of a unit.

The open dots on the x-axis indicate there is no rectangle with a length of 0 units or 10 units.

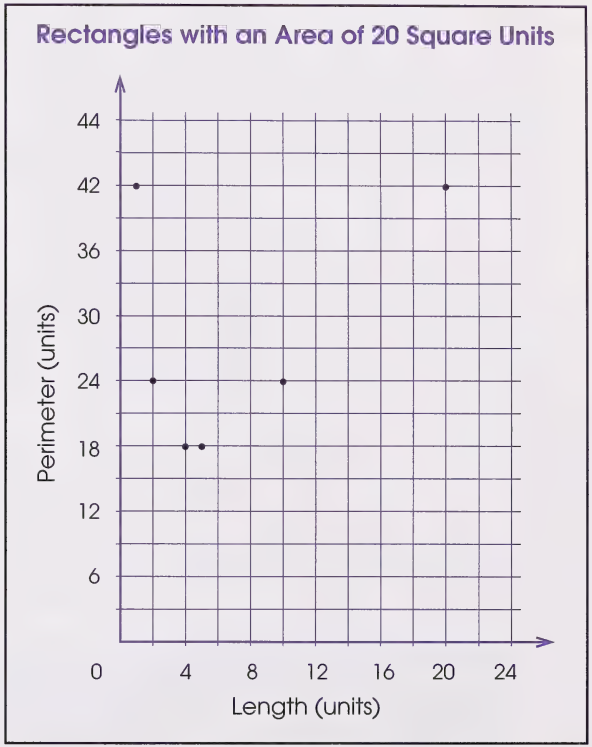
d. From the graph you can see that the area increases as the length increases. The maximum area is for the rectangle with a length of 5 units; this is the turning point. After that the area decreases as the length increases.

e. The rectangle that is 5 units by 5 units has the greatest area. This rectangle is a square.

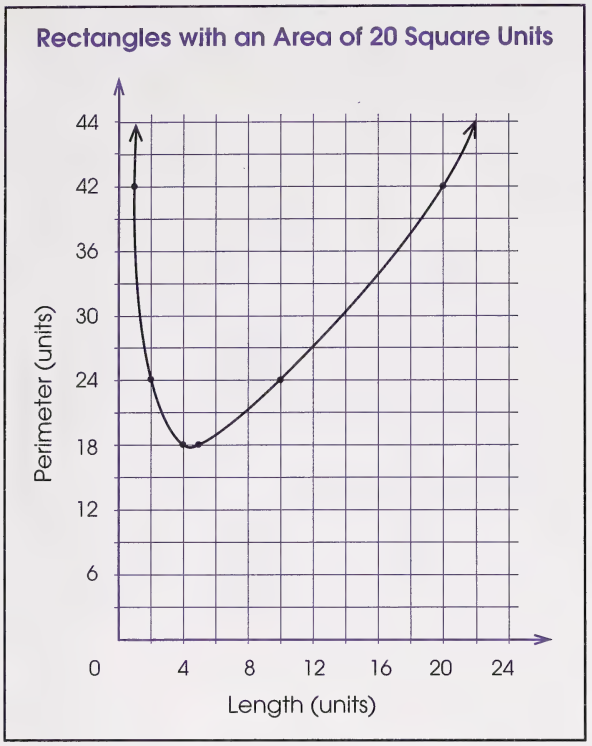
8. a.

Rectangles with an Area of 20 Square Units		
Length (units)	**Width (units)**	**Perimeter (units)**
1	20	42
2	10	24
4	5	18
5	4	18
10	2	24
20	1	42

b.



c.



The dots are joined because it is possible to have a length that is a fraction of a unit.

d. The perimeter decreases as the length of the rectangle increases. The least perimeter is for a rectangle with a length of about 4.5 units; this is the turning point. After that, the perimeter increases as the length increases.

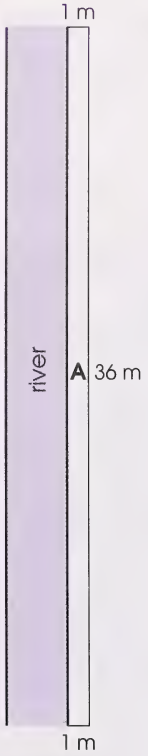
9. You may list the rectangles with an area of 144 m^2 to find the one with the least perimeter.

Rectangles with an Area of 144 m^2		
Length (m)	**Width (m)**	**Perimeter (m)**
1	144	290
2	72	148
3	48	102
4	36	80
6	24	60
8	18	52
9	16	50
12	12	48 ← minimum perimeter
16	9	50
18	8	52
24	6	60
36	4	80
48	3	102
72	2	148
144	1	290

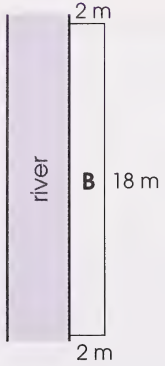
The rectangle with a length of 12 m and a width of 12 m has the least perimeter.

So, the dog run should be 12 m by 12 m.

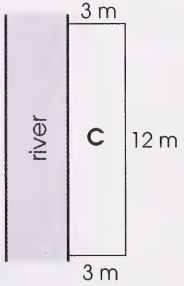
10. a.



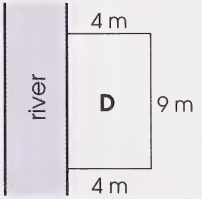
$1 + 1 + 36 = 38$ m



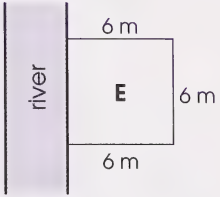
$2 + 2 + 18 = 22$ m



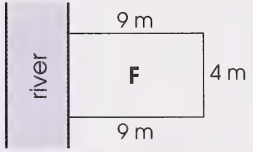
$12 + 3 + 3 = 18$ m



$4 + 4 + 9 = 17$ m

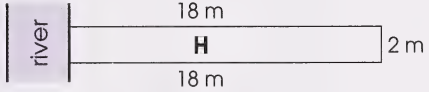


$6 + 6 + 6 = 18$ m

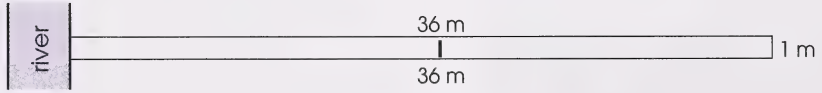


$9 + 9 + 4 = 22$ m

$12 + 12 + 3 = 27$ m



$18 + 18 + 2 = 38$ m



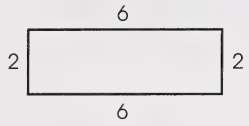
$36 + 36 + 1 = 73$ m

b. The most economical design is pen D.

c. The most expensive design is pen I.

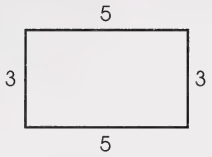
11. You can use the guess, check, and revise strategy to solve the problem. The pen will be 10 m by 11 m.

12. The following are different shapes of pens with 16 m of fencing.



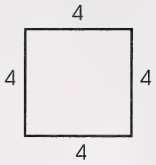
$A = 6 \times 2$

$= 12 \text{ m}^2$



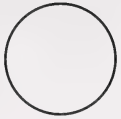
$A = 5 \times 3$

$= 15 \text{ m}^2$



$A = 4 \times 4$

$= 16 \text{ m}^2$



$C = 2\pi r$

$16 = 2\pi r$

$r = \dfrac{16}{2\pi}$

$\doteq 2.55$

$A = \pi r^2$

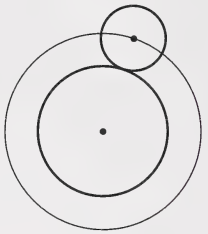
$\doteq \pi \times (2.55)^2$

$\doteq 20 \text{ m}^2$

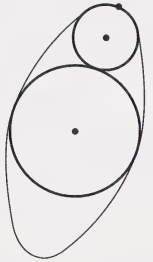
A circular pen gives the largest area.

Now Try This

13. a. All points traced by the centre of the small circle would lie on a circle around the larger circle.



b. The points traced by a dot on the small circle would be on an elliptical shape around the large circle.



Section 2: Activity 2

1. No, the right-side and left-side views are not the same.

2. You have to flip the left side over and then slide it onto the right side.

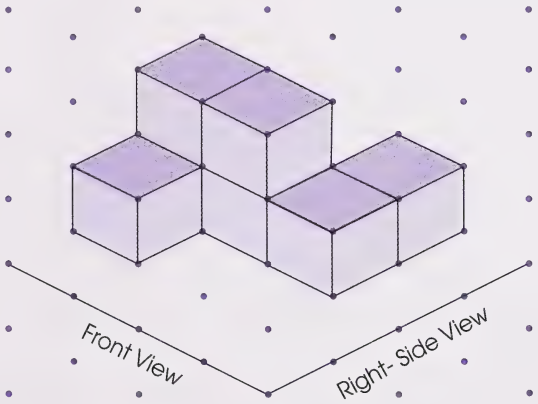
3. The figures labelled Right-Side View and Left-Side View are reflections of each other.

4. It may be difficult to visualize which are the front lines. You may need to erase hidden lines after you have completed your drawing.

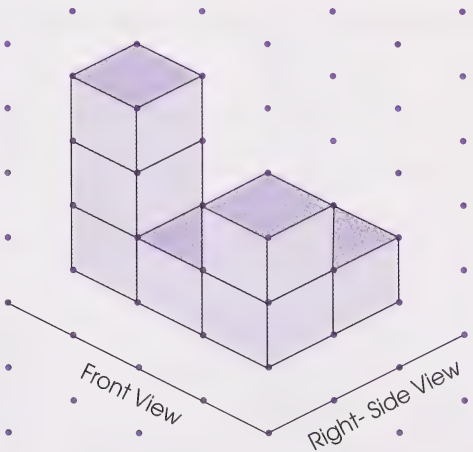
5. You should find that your ability to visualize and draw the different views of a 3-D model improves with practise.

6. The left-side view was not included in Example 1 because it is identical to the right-side view; therefore, it is not necessary.

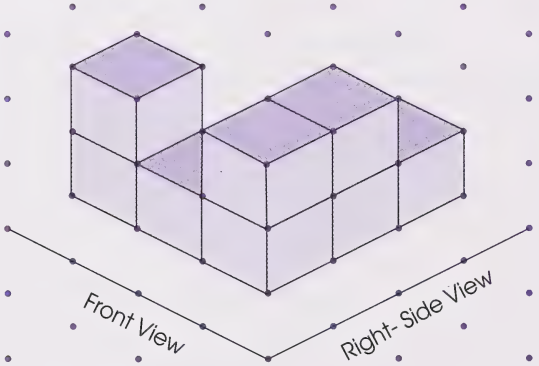
7.

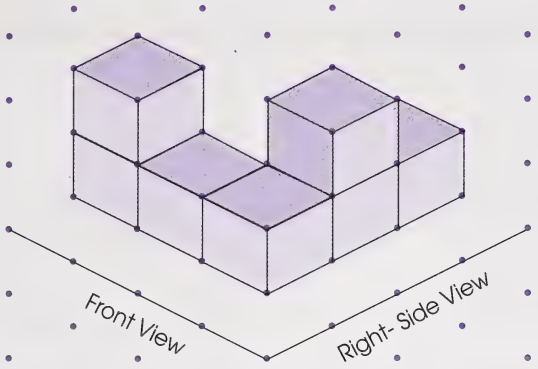


8. a.



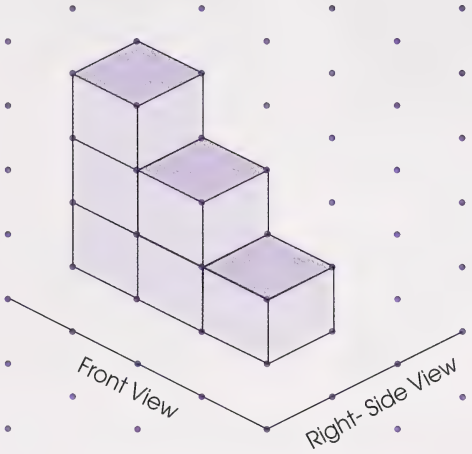
b. There are two possible drawings that satisfy the three views.



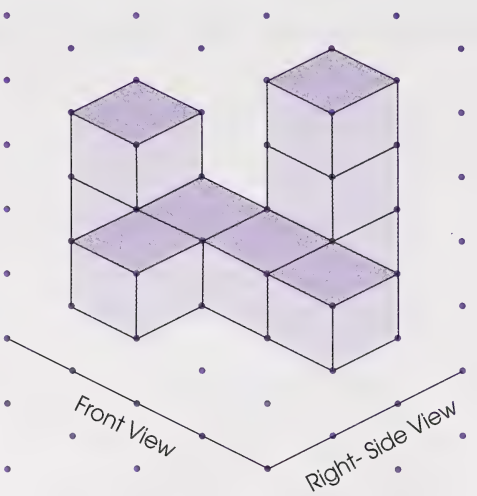


9. Your constructed models should look similar to the drawings in the answer to question 8.a. and 8.b.

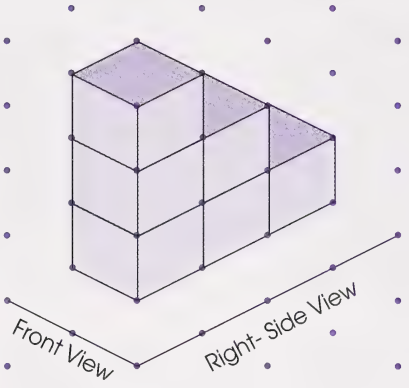
10. **a.**



b.



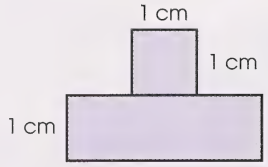
c.



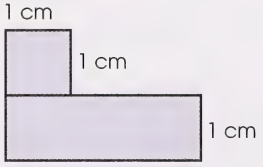
The drawing is basically the same as the answer to question 10.a. The only difference is that the side facing the front in the answer to question 10.a. is now facing the right side.

d. Yes, the model is correct for both drawings.

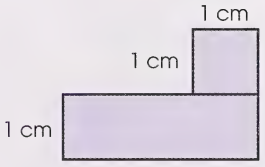
11. a.



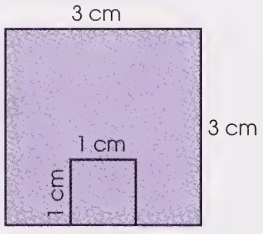
Front View



Right-Side View

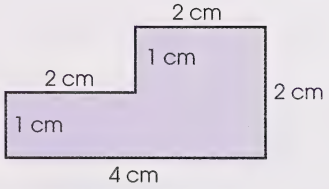


Left-Side View

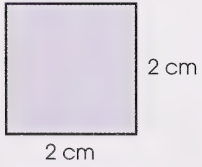


Top View

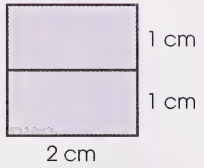
b.



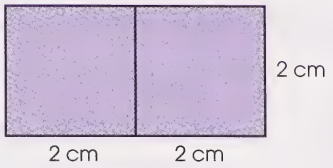
Front View



Right-Side View

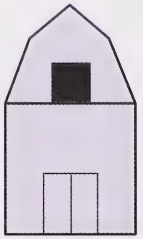


Left-Side View



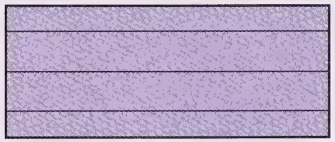
Top View

12.


Front View

Right-Side View

Left-Side View


Top View

Section 2: Activity 3

1. Yes, their heights are the same.

2. You need to repeat the procedure two more times.

3. The volume of the rectangular prism is three times larger than the volume of the rectangular pyramid.

$$\text{Volume of pyramid} = \frac{1}{3} \times \text{Volume of prism}$$
$$= \frac{1}{3}BH$$

4. a. $B = s^2$
$= (8)^2$
$= 64$

$V = \frac{1}{3}BH$
$= \frac{1}{3}(64)(20)$
$= 426.\overline{6}$

The volume is about $426.7\ \text{cm}^3$.

b. $B = \frac{1}{2}bh$
$= \frac{1}{2}(4.8)(7.2)$
$= 17.28$

$V = \frac{1}{3}BH$
$= \frac{1}{3}(17.28)(6.3)$
$= 36.288$

The volume is about $36.3\ \text{m}^3$.

c. $B = \frac{nsa}{2}$
$= \frac{6(4)(3.5)}{2}$
$= 42$

$V = \frac{1}{3}BH$
$= \frac{1}{3}(42)(7.2)$
$= 100.8$

The volume is $100.8\ \text{cm}^3$.

d. $B = \ell w$
$= (22.4)(13)$
$= 291.2$

$V = \frac{1}{3}BH$
$= \frac{1}{3}(291.2)(28.2)$
$= 2737.28$

The volume of the pyramid is about $2737.3\ \text{m}^3$.

5. a. $B = s^2$
$= (230)^2$
$= 52\,900$

$V = \frac{1}{3}BH$
$= \frac{1}{3}(52\,900)(147)$
$= 2\,592\,100$

The volume of stone in the pyramid is $2\,592\,100\ \text{m}^3$.

b. $2\,592\,100 \div 2\,300\,000 = 1.127$

Each block had an approximate volume of $1.127\ \text{m}^3$.

6. $B = s^2$

$= (21)^2$

$= 441$

$V = \frac{1}{3}BH$

$H = \frac{3V}{B}$

$= \frac{3(2013.9)}{441}$

$= 13.7$

The smaller pyramids are 13.7 m high.

7. $V = \frac{1}{3}BH$

$= \frac{1}{3}(11.4)(3.2)$

$= 12.16$

The tent holds about 12.2 m^3 of air.

8. **Method 1**

V of cube $= s^3$

$= (60)^3$

$= 216\,000$

V of each pyramid $= 216\,000 \div 6$

$= 36\,000$

The volume of each pyramid is 36 000 cm^3.

Method 2

$B = s^2$

$= (60)^2$

$= 3600$

$V = \frac{1}{3}BH$

$= \frac{1}{3}(3600)(30)$

$= 36\,000$

The volume of each pyramid is 36 000 cm^3.

9. Yes, the radius of the top of the cone is the same as the radius of the glass.

10. The cone must be filled and poured into the glass three times in order to fill the glass completely.

11. Divide the volume of the glass by 3.

12. $V = \frac{1}{3}\pi r^2 h$

13. **a.** $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(5)^2(12)$

$= \frac{1}{3}\pi(25)(12)$

$= 314.159\,265\,4$

The volume is about 314 cm^3.

b. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(4.5)^2(9)$

$= \frac{1}{3}\pi(20.25)(9)$

$= 190.851\,753\,7$

The volume is about 191 cm^3.

c. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(6)^2(14)$

$= \frac{1}{3}\pi(36)(14)$

$= 527.787\,565\,8$

The volume is about $528\ \text{cm}^3$.

d. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(9)^2(20)$

$= \frac{1}{3}\pi(81)(20)$

$= 1696.460\,033$

The volume is about $1696\ \text{cm}^3$.

14. $V = \frac{1}{3}\pi r^2 h$

$h = \frac{3V}{\pi r^2}$

$= \frac{3(314)}{78.5}$

$= 12$

Remember: The area of a circle is πr^2; therefore, $\pi r^2 = 78.5$.

The pile of sand is 12 m high.

15. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(4)^2(12)$

$= \frac{1}{3}\pi(16)(12)$

$= 201.061\,929\,8$

The cone can hold approximately $201\ \text{cm}^3$ of crushed ice.

16. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(3.5)^2(7.2)$

$= \frac{1}{3}\pi(12.25)(7.2)$

$= 92.362\,824\,02$

Each cup can hold approximately 92 mL $\left(92\ \text{cm}^3\right)$ of water.

17. a. $V = \pi r^2 h$

$= \pi(5)^2(10)$

$= \pi(25)(10)$

$= 785.398\,163\,4$

The cylinder has a capacity of approximately 785 mL.

b. $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(5)^2(10)$

$= \frac{1}{3}\pi(25)(10)$

$= 261.799\,387\,8$

The cone has a capacity of approximately 262 mL.

c. The cylinder fills about $\frac{3}{4}$ of the cube.

The cone fills about $\frac{1}{4}$ of the cube.

d. The rectangular pyramid would fill $\frac{1}{3}$ of the cube.

Therefore, the pyramid would fill more of the cube than the cone.

Now Try This

18. There is the same amount of water in the milk as there is milk in the water.

When you pour a cup of milk into the water, the mixture is $\frac{1}{4}$ milk and $\frac{3}{4}$ water. Then when you pour a cup of this mixture into the milk, you are adding only $\frac{3}{4}$ of a cup of water.

Therefore, container A is $\frac{1}{4}$ milk and container B is $\frac{3}{4} \div 3 = \frac{1}{4}$ water.

Section 2: Activity 4

1. $SA = 6s^2$
$= 6(4)^2$
$= 6(16)$
$= 96$

$V = \ell wh$
$= 4(4)(4)$
$= 64$

The surface area is $96\ \text{cm}^2$, and the volume is $64\ \text{cm}^3$.

2. a. $SA = 6s^2$
$= 6(8)^2$
$= 6(64)$
$= 384$

$V = \ell wh$
$= 8(8)(8)$
$= 512$

The surface area is $384\ \text{cm}^2$, and the volume is $512\ \text{cm}^3$.

b. $\frac{384}{96} = 4$

The surface area increases 4 times.

c. $\frac{512}{64} = 8$

The volume increases 8 times.

3. $4 = 2^2$ and $8 = 2^3$

4. a. $SA = 6s^2$
$= 6(12)^2$
$= 6(144)$
$= 864$

$V = \ell wh$
$= 12(12)(12)$
$= 1728$

The surface area is $864\ \text{cm}^2$, and the volume is $1728\ \text{cm}^3$.

b. $\frac{864}{96} = 9$

The surface area increases 9 times.

c. $\frac{1728}{64} = 27$

The volume increases 27 times.

5. $9 = 3^2$ and $27 = 3^3$

6.

Change in Dimensions	Change in Surface Area	Change in Volume
doubled	2^2	2^3
tripled	3^2	3^3
halved	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$
n times greater	n^2	n^3

7. Volume changes faster.

8. Surface area changes by the square of the change in dimensions. Volume changes by the cube of the change in dimensions.

9. Surface area involves a product of two dimensions, while volume involves a product of three dimensions.

10.
$$P = 2\ell + 2w \qquad SA = 2B + PH$$
$$= 2(30) + 2(8) \qquad = 2(30)(8) + 76(20)$$
$$= 60 + 16 \qquad = 480 + 1520$$
$$= 76 \qquad = 2000$$

$$V = \ell wh$$
$$= 30(8)(20)$$
$$= 4800$$

The surface area is $2000\ \text{cm}^2$, and the volume is $4800\ \text{cm}^3$.

11. a.
$$P = 2\ell + 2w \qquad SA = 2B + PH$$
$$= 2(60) + 2(16) \qquad = 2(60)(16) + 152(40)$$
$$= 120 + 32 \qquad = 1920 + 6080$$
$$= 152 \qquad = 8000$$

$$V = \ell wh$$
$$= 60(16)(40)$$
$$= 38\,400$$

The surface area is $8000\ \text{cm}^2$, and the volume is $38\,400\ \text{cm}^3$.

b. Yes, the relationship between doubling the dimensions and the change in surface area and the change in volume is the same as the relationship for a cube.

Surface Area	Volume
$\frac{8000}{2000} = 4$ or 2^2	$\frac{38\,400}{4800} = 8$ or 2^3

12. a. It takes four times more material to make the box, and it holds eight times more cereal.

b. It's cheaper for the manufacturer to package cereal in larger boxes. That's why buying larger quantities is cheaper for the consumer.

c. Cereal packages come in different sizes in order to serve the needs and desires of a variety of consumers (such as single people, small families, large families, and frequent shoppers).

13. a. The ratio is $3:1$ $\left(\frac{18}{6}, \frac{30}{10}, \frac{75}{25}\right)$.

b. Three times more reinforcing tape is needed for the larger box $(164 \text{ cm} \rightarrow 492 \text{ cm})$.

c. Nine times $\left(3^2\right)$ more material is needed to build the larger box $\left(920 \text{ cm}^2 \rightarrow 8280 \text{ cm}^2\right)$.

d. The volume of the larger box is 27 times $\left(3^3\right)$ bigger than the volume of the smaller box $\left(1500 \text{ cm}^3 \rightarrow 40\,500 \text{ cm}^3\right)$.

14. a. Based on previous calculations for surface area and volume, the surface area should be four times larger $\left(\text{or } 1608 \text{ cm}^2\right)$ and the volume should be eight times larger $\left(\text{or } 4824 \text{ cm}^3\right)$.

b. For a can with a radius of 8 cm and a height of 24 cm, the calculations for surface area and volume are as follows:

$$C = 2\pi r$$
$$= 2\pi(8)$$
$$= 50.265\,482\,46$$

$$SA = 2B + CH$$
$$\doteq 2(\pi)(8)^2 + 50.265(24)$$
$$\doteq 1608$$

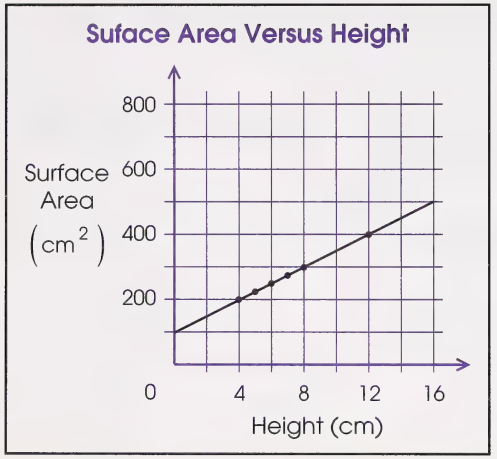
$$V = \pi r^2 h$$
$$= \pi(8)^2(24)$$
$$= \pi(64)(24)$$
$$\doteq 4825$$

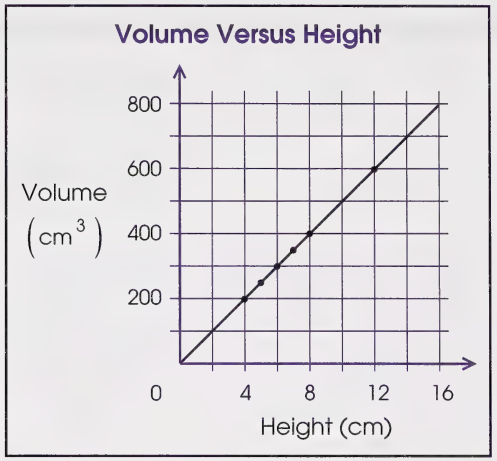
The calculated answers are the same or very close to the predicted answer.

15. a. Height versus surface area and volume for a can with a radius of 4 cm.

Height h (cm)	Surface Area $SA\ (cm^2)$	Volume $V\ (cm^3)$
4	201	201
5	226	251
6	251	302
7	276	352
8	302	402
12	402	603

b.





Volume changes more quickly than surface area when the height changes.

c. The surface area of a can with a height of 0 cm is approximately 100 cm^2. This is the surface area of the two ends of the can.

The volume of a can with a height of 0 cm is 0 cm^3. If there is no height, then there is no volume. You need three dimensions for volume.

16. a. **Cylinder**

$$SA = 2B + CH$$
$$= 2\pi(10)^2 + 2\pi(10)(20)$$
$$= 2\pi(100) + 2\pi(10)(20)$$
$$= 1884.955\,592$$

Cube

$$SA = 6s^2$$
$$= 6(18.45)^2$$
$$= 6(340.4025)$$
$$= 2042.415$$

The surface area of the can is about $1885\ \text{cm}^2$, and the surface area of the cube is $2042\ \text{cm}^2$.

b. The surface area of the cylinder is less than the surface area of the cube.

c. The cylinder uses less material to hold the same volume. Therefore, the packaging industry will try to use cylindrical containers where possible in order to save costs.

17. a.

Volume (cm^3)	Length (cm)	Width (cm)	Height (cm)	Surface Area (cm^2)
4000	10	10	40	1700
4000	15	15	17.778	1291.667
4000	20	20	10	1200
4000	19	19	11.080	1203.105
4000	21	21	9.070	1202.905

Note: The heights are determined by dividing the volume by the area of the base.

The most efficient size for a square-bottomed container holding $4000\ \text{cm}^3$ would be $20\ \text{cm} \times 20\ \text{cm} \times 10\ \text{cm}$.

b. In an earlier question, it was concluded that a cylinder has less surface area for a given volume than a cube. Therefore, a round-bottomed container may be more efficient.

The following trial and error calculations confirm that a round-bottomed container has a smaller surface area for a volume of $4000\ \text{cm}^3$.

Volume (cm^3)	Radius (cm)	Height (cm)	Surface Area (cm^2)
4000	5	50.930	1678.540
4000	10	12.732	1114.159
4000	15	5.659	1240.192
4000	9	15.719	1143.358
4000	11	10.523	1107.405

Note: The heights are determined by dividing the volume by the area of the base.

c. Other factors that may influence the size and shape of a container are as follows:

- ease of handling
- ease of storage before use
  i.e., a square-bottomed container may be folded flat before use

Now Try This

18. a. The next layer will be 25. Thus, the next pyramidal number is $30 + 25 = 55$.

b. The numbers that make up each row are called square numbers.

c. You would add the first ten square numbers.

Section 2: Follow-up Activities

Extra Help

1. a. $V = \ell wh$
$= 6(4)(6)$
$= 144 \text{ cm}^3$

b. $V = s^2 h$
$= (12)^2 (15)$
$= 144(15)$
$= 2160 \text{ cm}^3$

c. $V = \pi r^2 h$
$= \pi (4)^2 (6)$
$= \pi (16)(6)$
$\doteq 302 \text{ cm}^3$

2. a. $V = \frac{1}{3} \ell wh$
$= \frac{1}{3}(6)(4)(6)$
$= 48 \text{ cm}^3$

b. $V = \frac{1}{3} s^2 h$
$= \frac{1}{3}(12)^2 (15)$
$= \frac{1}{3}(144)(15)$
$= 720 \text{ cm}^3$

c. $V = \frac{1}{3} \pi r^2 h$
$= \frac{1}{3} \pi (4)^2 (6)$
$= \frac{1}{3} \pi (16)(6)$
$\doteq 101 \text{ cm}^3$

3. a. Yes, each pair has the same base and the same height.

b. The volumes of each pyramid or cone is one-third of the volume of its corresponding prism or cylinder.

4. a. $V = 200 \times 2^3$

$= 200 \times 8$

$= 1600 \text{ cm}^3$

b. $V = 48 \times \left(\frac{1}{2}\right)^3$

$= 48 \times \frac{1}{8}$

$= 6 \text{ cm}^3$

c. $V = 30 \times 3^3$

$= 30 \times 27$

$= 810 \text{ cm}^3$

d. $V = 256 \times \left(\frac{1}{4}\right)^3$

$= 256 \times \frac{1}{64}$

$= 4 \text{ cm}^3$

Enrichment

1. About $\frac{2}{3}$ of the can is filled with water.

2. The sphere has a volume equal to about two-thirds of the volume of the cylinder.

3. $V = \frac{2}{3}\pi r^2 h$, where $h = 2r$

$= \frac{2}{3}\pi r^2 (2r)$

$= \frac{4}{3}\pi r^3$

4. a. $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi (6)^3$

$= \frac{4}{3}\pi (216)$

$= 904.778\,684\,2$

The volume is about 905 cm^3.

b. $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi (4.2)^3$

$= \frac{4}{3}\pi (74.088)$

$= 310.339\,088\,7$

The volume is about 310 cm^3.

5. $V = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi (10.8)^3$

$= \frac{4}{3}\pi (1259.712)$

$= 5276.669\,286$

A ten-pin bowling ball contains approximately 5277 cm^3 of rubber.

6. a. The balls fill $\frac{2}{3}$ of the space in the cylinder.

b. **Container**

$$V = \pi r^2 h$$
$$= \pi (3.9)^2 (23.4)$$
$$= \pi (15.21)(23.4)$$
$$= 1118.136\,808$$

Tennis Balls

$$V = 3\left(\frac{4}{3}\pi r^3\right)$$
$$= 4\pi r^3$$
$$= 4\pi (3.9)^3$$
$$= 4\pi (59.319)$$
$$= 745.424\,538\,5$$

The volume of the container is about $1118\ \text{cm}^3$, and the volume of the tennis balls is about $745\ \text{cm}^3$.

c. $$\frac{V \text{ of balls}}{V \text{ of container}} \doteq \frac{745}{1118}$$
$$\doteq 0.67$$

Since $0.67 = \frac{2}{3}$, the prediction is correct.

7. a. The volume should increase about eight times.

b.
$$V = \frac{4}{3}\pi r^3 \qquad V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi (10)^3 \qquad = \frac{4}{3}\pi (20)^3$$
$$= \frac{4}{3}\pi (1000) \qquad = \frac{4}{3}\pi (8000)$$
$$= 4188.790\,205 \qquad = 33\,510.321\,64$$

The volume of the balloon with a 10-cm radius is about $4189\ \text{cm}^3$, and the volume of the balloon with a 20-cm radius is about $33\,510\ \text{cm}^3$.

$$\frac{V \text{ of 20-cm balloon}}{V \text{ of 10-cm balloon}} \doteq \frac{33\,510}{4189}$$
$$\doteq 8$$

Therefore, the prediction is correct.

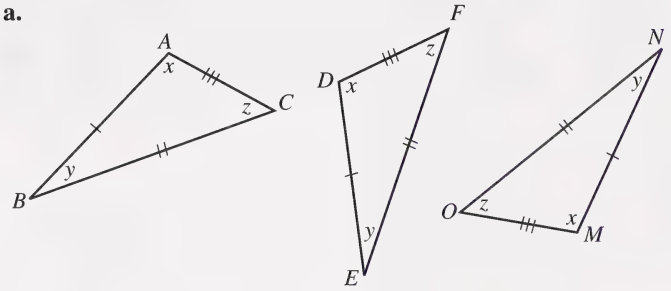
Section 3: Activity 1

1. Yes, the two figures coincide exactly.

2. Yes, the two figures appear to be the same size and shape.

3. You would get four figures that appear to be the same size and shape.

4. a.



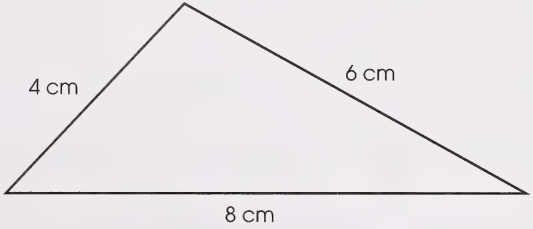
b. Refer to the answer to question 4.a.

c. Refer to the answer to question 4.a.

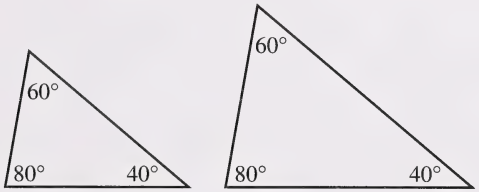
d. Refer to the answer to question 4.a.

e. Yes, the three triangles coincide exactly.

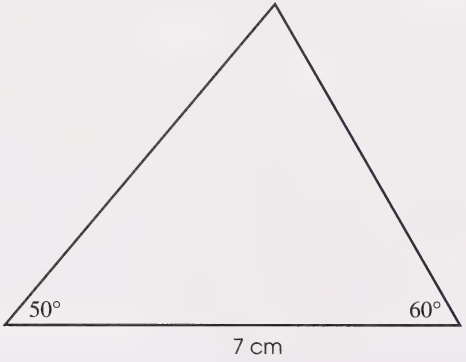
5. **Property 1:** They will all look like this example.



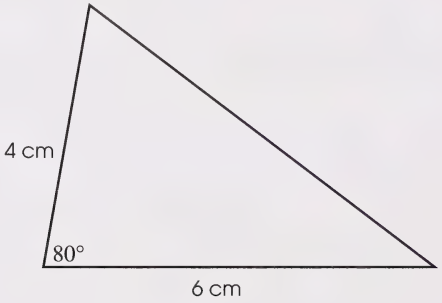
Property 2: They are all similar in shape but could be different in size.



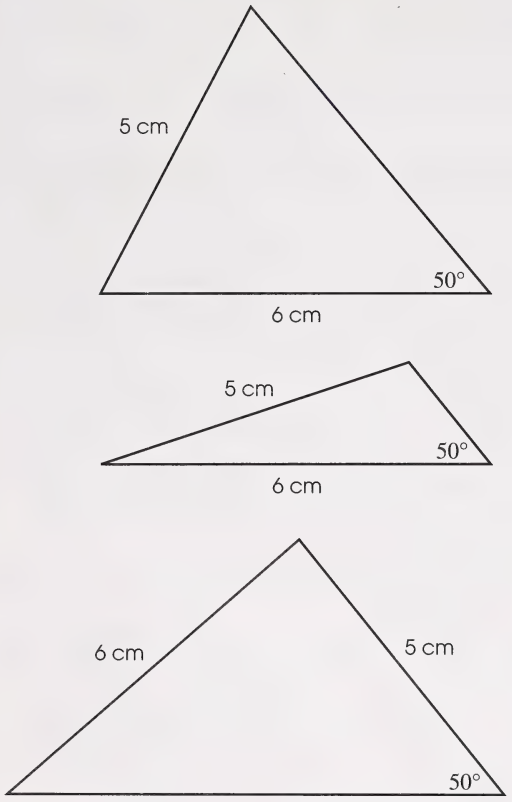
Property 3: They will all look like this example.



Property 4: They will all look like this example.



Property 5: The triangles will look like one of these three examples.



6. **Property 1:** Yes
   Property 2: No
   Property 3: Yes
   Property 4: Yes
   Property 5: No

7. Two triangles are congruent under the following conditions.

   SSS The three corresponding sides are equal.

   ASA Two angles and one side of a triangle are equal to the corresponding two angles and one side. (It does not have to be the contained side. If two angles are equal, then the third angle must also be equal.)

   SAS Two sides and the contained angle of a triangle are equal to the corresponding two sides and contained angle.

8. **a.** SSS
   $\Delta ABC \cong \Delta XYZ$

   b. SAS
   $\Delta CAT \cong \Delta DOG$

9. **a.** $\angle A = \angle X$
   $\angle B = \angle Y$
   $\angle C = \angle Z$

   $AB = XY$
   $BC = YZ$
   $AC = XZ$

   b. $\angle C = \angle D$
   $\angle A = \angle O$
   $\angle T = \angle G$

   $CA = DO$
   $AT = OG$
   $CT = DG$

10. **a.** Either $\angle T = \angle N$ or $PO = MA$

b. $UY = WO$ or $BY = NO$ or $BU = NW$

c. Either $NT = PN$ or $\angle O = \angle A$

11. **a.**

$BC = 32$ cm	$\angle C = 42°$
$DF = 12$ cm	$\angle B = 18°$
$DE = 24$ cm	$\angle E = 18°$

b.

$BO = 2.4$ cm	$\angle Y = 37°$
$OY = 3.5$ cm	$\angle M = 53°$
$MN = 4.2$ cm	$\angle A = 90°$
	$\angle N = 37°$

12. $\Delta ABC \cong \Delta XYZ$ (ASA or AAS)
$\Delta DEF \cong \Delta LKM$ (SSS)
$\Delta NOP \cong \Delta GHJ$ (SAS)
$\Delta TWV \cong \Delta QSR$ (ASA)

13. **a.** $\angle ACB = \angle ACD$ because the other two angles in the two triangles are equal. That is,
$\angle ACB = 180 - (x + y) = \angle ACD$.

b. Yes, AC is common to both triangles; so, $\Delta ABC \cong \Delta ADC$ (ASA or SAA).

14.

$\angle P = \angle M$	$PI = MA$
$\angle I = \angle A$	$IN = AT$
$\angle N = \angle T$	$PN = MT$

15. **a.**
$$\begin{aligned} 3x + 40 &= 5x + 10 \\ 3x + 40 - 40 &= 5x + 10 - 40 \\ 3x &= 5x - 30 \\ 3x - 5x &= 5x - 30 - 5x \\ -2x &= -30 \\ \frac{-2x}{-2} &= \frac{-30}{-2} \\ x &= 15 \end{aligned}$$

$$\begin{aligned} \therefore \angle T &= 3x + 40 \\ &= 3(15) + 40 \\ &= 45 + 40 \\ &= 85 \end{aligned}$$

The measure of $\angle T$ is 85°.

b.
$$\begin{aligned} 2y - 8 &= y - 2 \\ 2y - 8 + 8 &= y - 2 + 8 \\ 2y &= y + 6 \\ 2y - y &= y + 6 - y \\ y &= 6 \end{aligned}$$

$\therefore OY = 2y - 8$

$= 2(6) - 8$

$= 12 - 8$

$= 4$

The measure of OY is 4 cm.

c. Since $TY = FN$ and $y = 6$, then TY can be calculated as follows:

$3y - 14.5 = 3(6) - 14.5$

$= 18 - 14.5$

$= 3.5$

The measure of TY is 3.5 cm.

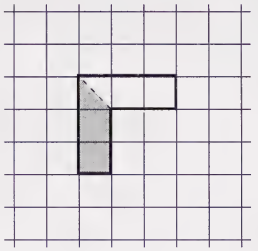
16. a. Two triangles with the same perimeter are sometimes congruent.

b. Two triangles with the same area are sometimes congruent.

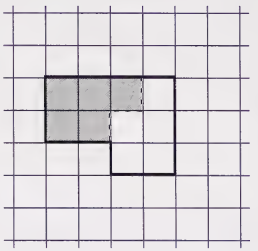
c. Two squares with the same perimeter are always congruent. Two squares with the same area are always congruent. This is true because the angles in a square are always the same measure and the side are always the same length.

Now Try This

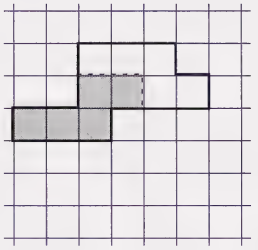
17. a.



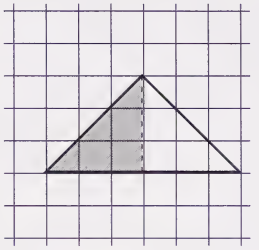
b.



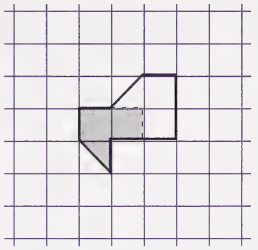
c.



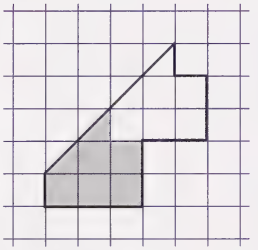
d.



e.



f.



18. **a.** reflection

b. reflection

c. reflection and translation

d. reflection

e. rotation

f. translation

Section 3: Activity 2

1. $\angle A = \angle R = 37°$
$\angle B = \angle S = 53°$
$\angle C = \angle T = 90°$

2. **a.** $\frac{AC}{AB} = \frac{4}{5}$ $\quad \frac{RT}{RS} = \frac{8}{10} = \frac{4}{5}$

$\frac{BC}{AB} = \frac{3}{5}$ $\quad \frac{ST}{RS} = \frac{6}{10} = \frac{3}{5}$

$\frac{BC}{AC} = \frac{3}{4}$ $\quad \frac{ST}{RT} = \frac{6}{8} = \frac{3}{4}$

b. The ratio of any two sides of ΔABC are the same as the ratio of the two corresponding sides of ΔRST.

3. **a.** $\frac{DE}{XY} = \frac{6}{9} = \frac{2}{3}$ $\quad \frac{DF}{XZ} = \frac{3}{4.5} = \frac{2}{3}$ $\quad \frac{EF}{YZ} = \frac{5}{7.5} = \frac{2}{3}$

The ratios of the corresponding sides are equal.

b. **i.** $\frac{DE}{DF} = \frac{6}{3} = \frac{2}{1}$ $\quad \frac{XY}{XZ} = \frac{9}{4.5} = \frac{2}{1}$

ii. $\frac{DF}{EF} = \frac{3}{5}$ $\quad \frac{XZ}{YZ} = \frac{4.5}{7.5} = \frac{3}{5}$

c. They are pairs of equivalent ratios because you are comparing ratios of corresponding sides.

4. **a.** $180 - (40 + 60) = 80°$

$\therefore \angle C = 80°$ and $\angle F = 80°$

b. Yes, $\Delta ABC \sim \Delta DEF$ because their corresponding angles have the same measure.

5. **a.** $\frac{24}{36} = \frac{30}{45} = \frac{52}{78}$

Yes, these triangles are similar since the corresponding sides are proportional.

b. Not enough information is given to determine if these triangles are similar.

c. Yes, these triangles are similar since all pairs of corresponding angles are equal.

d. $\frac{5}{2.2} \neq \frac{3.5}{2.2}$

All the sides of one triangle are not proportional to the corresponding sides of the other triangle.

6. The following is one way to solve this question.

$$\frac{DE}{AB} = \frac{EF}{BC} \qquad \frac{DE}{AB} = \frac{DF}{AC}$$
$$\frac{5}{2} = \frac{3.2}{a} \qquad \frac{5}{2} = \frac{6}{b}$$
$$5 \times a = 3.2 \times 2 \qquad 5 \times b = 6 \times 2$$
$$5a = 6.4 \qquad 5b = 12$$
$$\frac{5a}{5} = \frac{6.4}{5} \qquad \frac{5b}{5} = \frac{12}{5}$$
$$a = 1.28 \text{ cm} \qquad b = 2.4 \text{ cm}$$

7. The following is one way to solve this question.

$$\frac{RS}{XY} = \frac{ST}{YZ} \qquad \frac{RS}{XY} = \frac{RT}{XZ}$$
$$\frac{4}{2.5} = \frac{r}{1.5} \qquad \frac{4}{2.5} = \frac{5}{y}$$
$$4 \times 1.5 = r \times 2.5 \qquad 4 \times y = 5 \times 2.5$$
$$6 = 2.5r \qquad 4y = 12.5$$
$$2.5r = 6 \qquad \frac{4y}{4} = \frac{12.5}{4}$$
$$\frac{2.5r}{2.5} = \frac{6}{2.5} \qquad y = 3.125 \text{ cm}$$
$$r = 2.4 \text{ cm}$$

8. Yes. The corresponding angles are congruent, and the corresponding sides are proportional.

9. No. The corresponding angles are congruent; but the corresponding sides are proportional, not equal.

10.
$$\frac{8}{10} = \frac{J}{2.2}$$
$$8 \times 2.2 = J \times 10$$
$$17.6 = 10J$$
$$10J = 17.6$$
$$\frac{10J}{10} = \frac{17.6}{10}$$
$$J = 1.76 \text{ m}$$

Joanna is 1.76 m (or 176 cm) tall.

11.

$$\frac{AB}{AD} = \frac{BC}{DE}$$

$$\frac{6}{x+6} = \frac{8}{26}$$

$$6 \times 26 = 8 \times (x+6)$$

$$156 = 8x + 48$$

$$8x + 48 = 156$$

$$8x + 48 - 48 = 156 - 48$$

$$8x = 108$$

$$\frac{8x}{8} = \frac{108}{8}$$

$$x = 13.5 \text{ cm}$$

12. Yes. The angles are always 90°, and the corresponding sides are always proportional.

13. **a.** All three triangles share a 30° angle, and each has a 90° angle; therefore, the remaining angle in these triangles is 60°. Thus, the three triangles are similar because their corresponding angles are congruent.

b. ΔADJ, ΔAEK, ΔAFL, and ΔAGM

c. $BH = 1$ cm and $AB = 2$ cm

$$\frac{BH}{AB} = \frac{1}{2}$$

d. $CI = 1.5$ cm and $AC = 3$ cm

$$\frac{CI}{AC} = \frac{1.5}{3}$$

$$= \frac{1}{2}$$

e. The ratios are all $\frac{1}{2}$ because they are corresponding sides of similar triangles, and therefore, must be proportional (have the same ratio).

f.

$$\frac{XY}{AX} = \frac{1}{2}$$

$$\frac{9.5}{AX} = \frac{1}{2}$$

$$9.5 \times 2 = 1 \times AX$$

$$19 = AX$$

$$AX = 19 \text{ cm}$$

g. The ratio of the length of the side opposite the 30° angle to the length of the side opposite the 90° angle is $\frac{1}{2}$.

14. **a.**

$$\frac{EB}{AE} = \frac{EB}{600} = \frac{1}{2}$$

$$EB = \frac{600}{2}$$

$$= 300$$

They are 300 m from the riverbank.

b. The river is 600 m wide.

c. AC is the distance that the canoe floats downstream while crossing the river. To find AC, you need to find AB first.

$$(AB)^2 + (EB)^2 = (AE)^2$$
$$(AB)^2 + 300^2 = 600^2$$
$$(AB)^2 + 90\,000 = 360\,000$$
$$(AB)^2 + 90\,000 - 90\,000 = 360\,000 - 90\,000$$
$$(AB)^2 = 270\,000$$
$$AB = \sqrt{270\,000}$$
$$\doteq 520$$

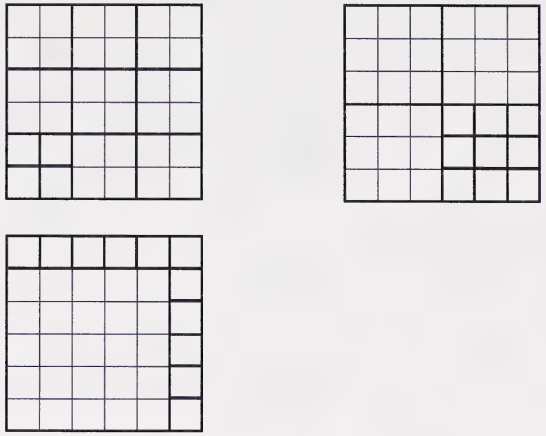
Now find AC.

$$\frac{AB}{AC} = \frac{EB}{DC}$$
$$\frac{520}{AC} \doteq \frac{300}{600}$$
$$520 \times 600 \doteq 300 \times AC$$
$$312\,000 \doteq 300\,AC$$
$$300\,AC \doteq 312\,000$$
$$\frac{300\,AC}{300} \doteq \frac{312\,000}{300}$$
$$AC \doteq 1040$$

They should begin their crossing about 1040 m upstream from that point.

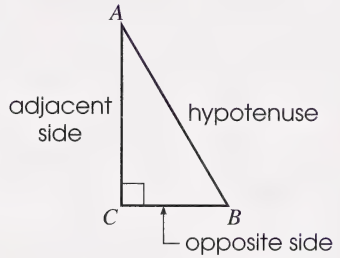
Now Try This

15. Answers will vary. Three possible solutions are given.

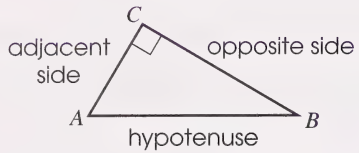


Section 3: Activity 3

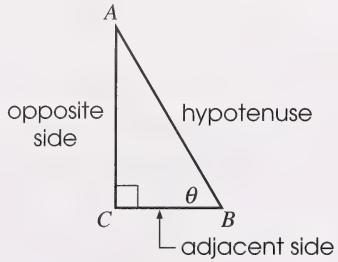
1. a.



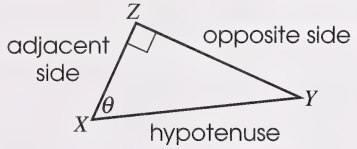
b.



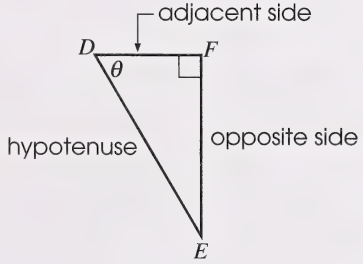
2. a.



b.



c.

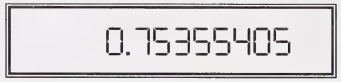


3. $\frac{BC}{AC} = \frac{3}{4}$ $= 0.75$ $\quad \frac{DE}{AE} = \frac{6}{8}$ $= 0.75$ $\quad \frac{FG}{AG} = \frac{9}{12}$ $= 0.75$

4. Yes, the ratio is the same for each triangle.

5. The measure of $\angle A$ is 37°, to the nearest degree.

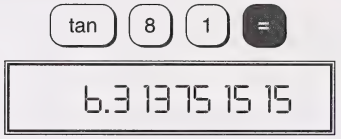
6.



This value, rounded to four decimal places, is 0.7536.

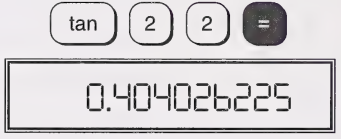
7. This value is close to the value of the tangent ratio that was obtained previously.

8. a.



$\therefore$ tan 81° ≐ 6.3138

b.



$\therefore$ tan 22° = 0.4040

c. tan 4 5 =

$\therefore \tan 45° = 1.0000$

d. tan 2 =

$\therefore \tan 2° = 0.0349$

9. a. $\tan \theta = \frac{12}{5}$ or 2.4000 $\quad \tan \alpha = \frac{5}{12}$ or 0.4167

b. $\tan \theta = \frac{36}{27}$ $\quad \tan \alpha = \frac{27}{36}$

$= \frac{4}{3}$ or 1.3333 $\quad = \frac{3}{4}$ or 0.7500

10. a. Shift tan • 3 9 8 5 =

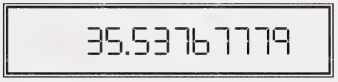
$\therefore \angle\theta \doteq 22°$

b. Shift tan 9 • 8 4 7 6 =

$\therefore \angle\theta \doteq 84°$

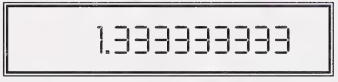
c. 5 ÷ 7 =

Shift tan =

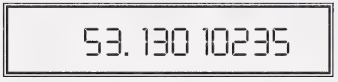


$\therefore \angle\theta \doteq 36°$

d. 4 ÷ 3 =



Shift tan =

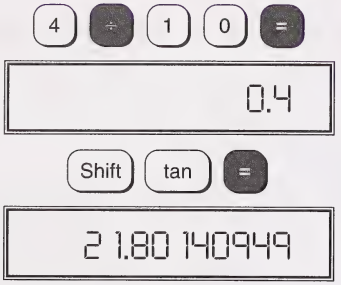


$\therefore \angle\theta \doteq 53°$

11. $\tan \angle D = \dfrac{\text{opp}}{\text{adj}}$

$= \dfrac{EF}{DE}$

$= \dfrac{4}{10}$

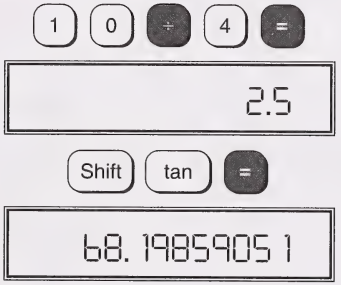


$\therefore \angle D \doteq 22°$

$\tan \angle F = \dfrac{\text{opp}}{\text{adj}}$

$= \dfrac{DE}{EF}$

$= \dfrac{10}{4}$

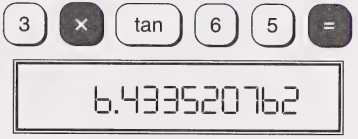


$\therefore \angle F \doteq 68°$

12. a. $\tan 65° = \dfrac{\text{opp}}{\text{adj}}$

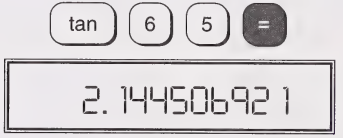
$\tan 65° = \dfrac{XY}{3}$

$XY = 3 \times \tan 65°$



Therefore, the length of XY is about 6.4 m.

b.



$\therefore \tan 65° \doteq 2.1445$

c. Method 1

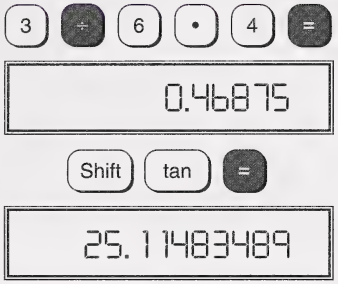
$\angle X = 90° - 65°$

$= 25°$

Method 2

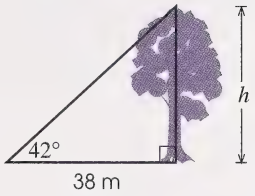
$$\tan \angle X = \frac{\text{opp}}{\text{adj}}$$

$$\doteq \frac{3}{6.4}$$



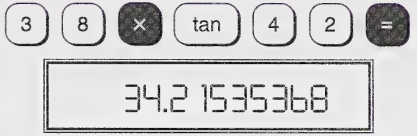
$\therefore \angle X \doteq 25°$

13.



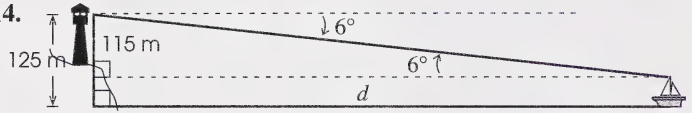
$$\frac{h}{38} = \tan 42°$$

$$h = 38 \times \tan 42°$$



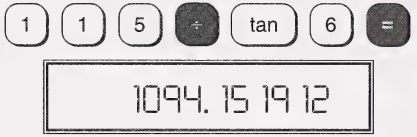
Therefore, the tree is about 34 m high.

14.



$$\tan 6° = \frac{115}{d}$$

$$d = \frac{115}{\tan 6°}$$

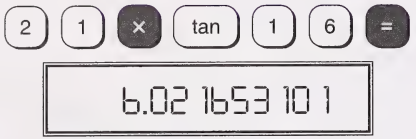


Therefore, the sailboat is about 1094 m (or 1.09 km) from shore.

15. $\tan 16° = \dfrac{\text{opp}}{\text{adj}}$

$\tan 16° = \dfrac{h}{21}$

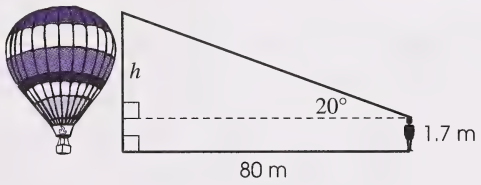
$h = 21 \times \tan 16°$



Therefore, the tree is about $6.0 + 1.6 = 7.6$ m tall.

16. A reasonable answer may range from a few metres to 100 m or more.

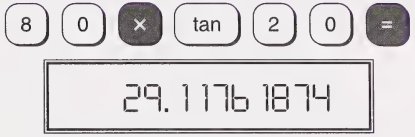
17.



$\tan \theta = \dfrac{h}{d}$

$\tan 20° = \dfrac{h}{80}$

$h = 80 \times \tan 20°$

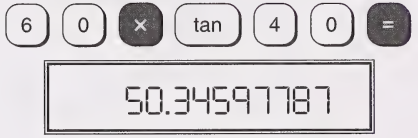


Therefore, the height of the inflated hot-air balloon is about $29.1 + 1.7 = 30.8$ m.

18. $\tan \theta = \dfrac{h}{d}$

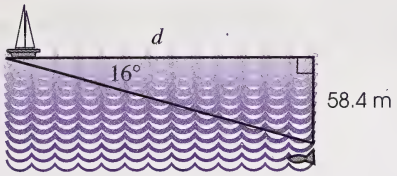
$\tan 40° = \dfrac{h}{60}$

$h = 60 \times \tan 40°$



Therefore, the height of the building is about 50.3 m.

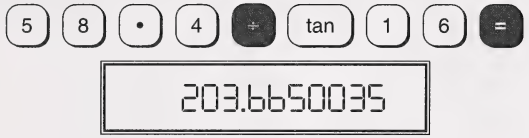
19.



$$\tan\theta = \frac{h}{d}$$

$$\tan 16° = \frac{58.4}{d}$$

$$d = \frac{58.4}{\tan 16°}$$



Therefore, the captain must move his schooner about 204 m in order to be directly over the school of fish.

20. a.

Angle	Tangent	Angle	Tangent
5°	0.0875	50°	1.1918
10°	0.1763	55°	1.4281
15°	0.2679	60°	1.7321
20°	0.3640	65°	2.1445
25°	0.4663	70°	2.7475
30°	0.5774	75°	3.7321
35°	0.7002	80°	5.6713
40°	0.8391	85°	11.4301
45°	1.000		

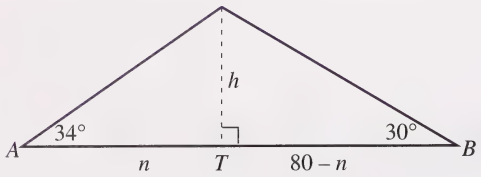
b. The value of the tangent ratio goes from close to 0 when the angle is 5° to over 11 when the angle is 85°.

c. From the pattern in question 20.a., it appears the tangent value for 0° should be 0. Using the calculator, the tangent of 0° is 0.

d. From the pattern in question 20.a., the tangent of 90° should be a very large number. When using a calculator, an error message for the value of tan 90° appears. As the angle gets close to 90° the length of the side you are dividing by is getting close to 0. At 90° the length of that side of the right triangle would be 0, and division by zero gives an error message.

Now Try This

21.



Let h be the height of the tree and n be the distance from A to the tree. Thus, $80 - n$ is the distance from B to the tree.

$$\tan\theta = \frac{h}{AT} \qquad \tan\theta = \frac{h}{TB}$$

$$\tan 34° = \frac{h}{n} \qquad \tan 30° = \frac{h}{80-n}$$

$$h = n \times \tan 34° \qquad h = (80-n) \times \tan 30°$$

$$\doteq 0.6745n \qquad \doteq (80-n)(0.5774)$$

$$\doteq 46.1880 - 0.5774n$$

Since $h = h$, then

$$0.6745n \doteq 46.1880 - 0.5774n$$

$$0.6745n + 0.5774n \doteq 46.1880 - 0.5774n + 0.5774n$$

$$1.2519n \doteq 46.1880$$

$$\frac{1.2519n}{1.2519} \doteq \frac{46.1880}{1.2519}$$

$$n \doteq 37$$

Substitute $n \doteq 37$ into one of the equations of h.

$$0.6745n \doteq 0.6745(37)$$

$$\doteq 25$$

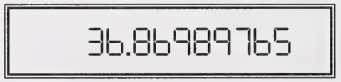
The tree is about 25 m high.

22. a. $\frac{BC}{AB} = \frac{3}{5}$ or 0.6 $\qquad \frac{DE}{AD} = \frac{6}{10}$ or 0.6

$\frac{FG}{AF} = \frac{9}{12}$ or 0.6

b. Yes, the ratios are the same for each triangle.

23.



This is 37°, rounded to the nearest degree. This value is $\angle A$.

24. a. $\frac{AC}{AB} = \frac{4}{5}$ or 0.8 $\qquad \frac{AE}{AD} = \frac{8}{10}$ or 0.8

$\frac{AG}{AF} = \frac{12}{15}$ or 0.8

b. Yes, the ratios are the same for each triangle.

25. 4 ÷ 5 = Shift cos =

Rounded to the nearest degree, the resulting angle is 37°.

26. The measure of $\angle A$ is the same as the values obtained using the tangent and sine ratios.

27. **a.** sin 3 0 =

$\therefore$ sin 30° = 0.5000

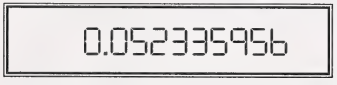
cos 3 0 =

$\therefore$ cos 30° = 0.8660

b. sin 8 7 =

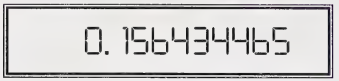
$\therefore$ sin 87° = 0.9986

cos 8 7 =



$\therefore$ cos 87° = 0.0523

c. sin 9 =

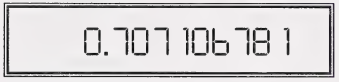


$\therefore$ sin 9° = 0.1564

cos 9 =

$\therefore$ cos 9° = 0.9877

d. sin 4 5 =



$\therefore$ sin 45° = 0.7071

cos 4 5 =

$\therefore$ cos 45° = 0.7071

e. sin 6 0 =

$\therefore$ sin 60° = 0.8660

cos 6 0 =

$\therefore$ cos 60° = 0.5000

28. a. $\sin 0° = 0$, $\sin 90° = 1$, $\cos 0° = 1$, and $\cos 90° = 0$.

b. The sine and cosine ratios are the same when the angle is 45°.

29. a. Shift sin • 8 2 =

55.08479375

$\therefore \angle C \doteq 55°$

b. Shift cos • 0 8 1 4 =

85.33095769

$\therefore \angle E \doteq 85°$

c. 5 ÷ 8 = Shift sin =

38.682 18745

$\therefore \angle A \doteq 39°$

d. 1 6 ÷ 2 3 = Shift cos =

45.920790 15

$\therefore \angle B \doteq 46°$

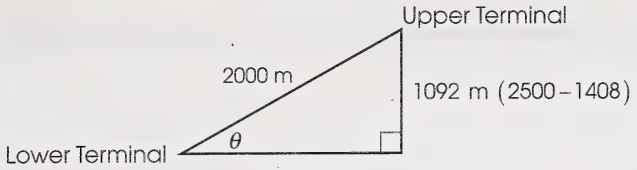
30. a.

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \cos A = \frac{\text{adj}}{\text{hyp}}$$
$$= \frac{30}{34} \qquad = \frac{16}{34}$$
$$\doteq 0.8824 \qquad \doteq 0.4706$$

$$\sin B = \frac{\text{opp}}{\text{hyp}} \qquad \cos B = \frac{\text{adj}}{\text{hyp}}$$
$$= \frac{16}{34} \qquad = \frac{30}{34}$$
$$\doteq 0.4706 \qquad \doteq 0.8824$$

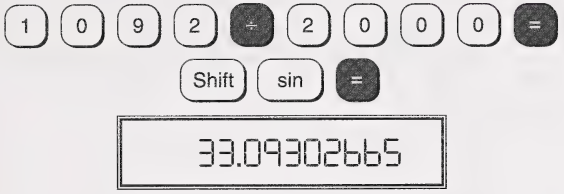
b. The ratios for $\sin A$ and $\cos B$ are the same. The ratios for $\sin B$ and $\cos A$ are also the same.

31.



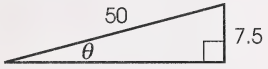
$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin\theta = \frac{1092}{2000}$$



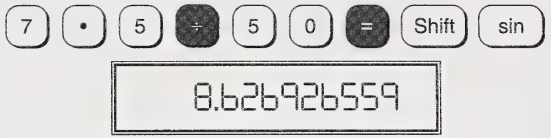
Therefore, the angle of inclination is about 33°.

32.



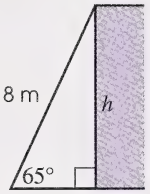
$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin\theta = \frac{7.5}{50}$$



Therefore, the airplane is climbing at an angle of about 9°.

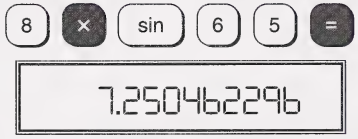
33.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

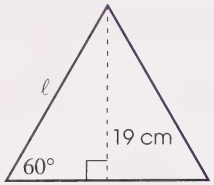
$$\sin 65° = \frac{h}{8}$$

$$h = 8 \times \sin 65°$$



Therefore, the ladder will reach about 7.25 m up the wall.

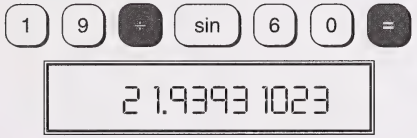
34.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 60° = \frac{19}{\ell}$$

$$\ell = \frac{19}{\sin 60°}$$

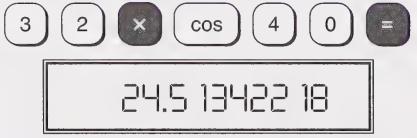


Therefore, each side is approximately 22 cm long.

35. $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$

$$\cos 40° = \frac{a}{32}$$

$$a = 32 \times \cos 40°$$



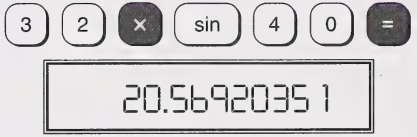
$\therefore\ a \doteq 25$ cm

You must find the measure of x before determining the value of b.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 40° = \frac{x}{32}$$

$$x = 32 \times \sin 40°$$



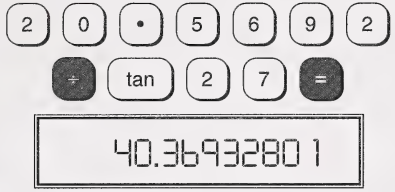
$\therefore\ x \doteq 20.5692$

Now find b.

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

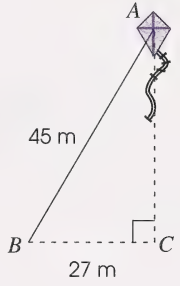
$$\tan 27° \doteq \frac{20.5692}{b}$$

$$b \doteq \frac{20.5692}{\tan 27°}$$



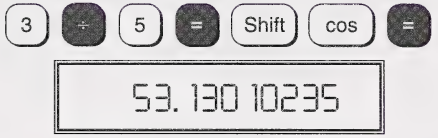
Therefore, the length of side a is about 25 cm, and the length of side b is about 40 cm.

36. Draw a diagram.



a. $\cos B = \frac{27}{45}$

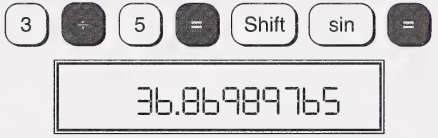
$= \frac{3}{5}$



Therefore, the angle the kite's string forms with the ground is about 53°.

b. $\sin A = \frac{27}{45}$

$= \frac{3}{5}$



Therefore, the angle the kite's string forms with a vertical line from the kite to the ground is about 37°.

Note: You may also use $90° - 53° = 37°$ to solve question 36.b.

c. You could have used one of several methods to determine the height of the kite. One method is given.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (27)^2 + b^2 &= (45)^2 \\ 729 + b^2 &= 2025 \\ 729 + b^2 - 729 &= 2025 - 729 \\ b^2 &= 1296 \\ b &= \sqrt{1296} \\ &= 36 \end{aligned}$$

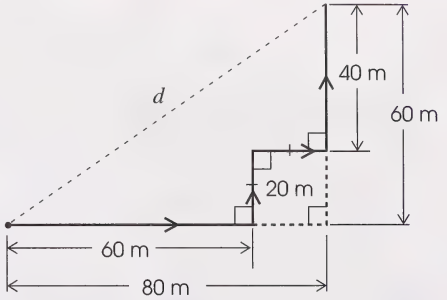
The kite is flying 36 m above the ground.

d. Answers will vary depending what method you used in question 36.c. Methods to determine the height of the kite include the following:

- Pythagorean Theorem (as used in 36.c.)
- tangent of angle B with the height as the opposite side and 27 m as the adjacent side
- sine of angle B with the height as the opposite side and the hypotenuse as the other side
- tangent of angle A with 27 m as the opposite side and the height as the adjacent side
- cosine of angle A with the height as the adjacent side and the hypotenuse as the other side

Now Try This

37. Draw a diagram to help illustrate the problem.



Use the Pythagorean Theorem to solve this problem.

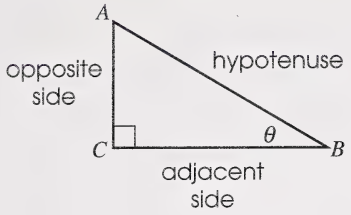
$$\begin{aligned} d^2 &= (80)^2 + (60)^2 \\ d^2 &= 6400 + 3600 \\ d^2 &= 10\,000 \\ d &= \sqrt{10\,000} \\ &= 100 \end{aligned}$$

Shawna is 100 m from her starting point.

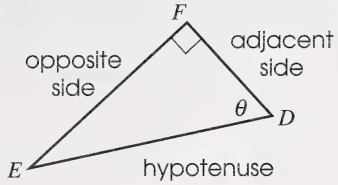
Section 3: Follow-up Activities

Extra Help

1. a.



b.



2. a. $\sin\theta = \frac{b}{c}$ $\qquad$ $\sin\alpha = \frac{a}{c}$

$\cos\theta = \frac{a}{c}$ $\qquad$ $\cos\alpha = \frac{b}{c}$

$\tan\theta = \frac{b}{a}$ $\qquad$ $\tan\alpha = \frac{a}{b}$

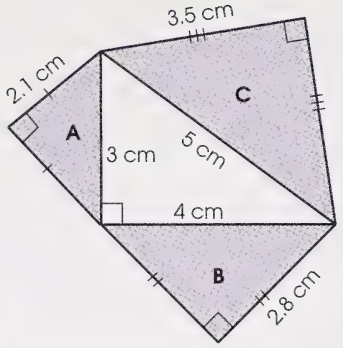
b. $\sin\theta = \frac{e}{f}$ $\qquad$ $\sin\alpha = \frac{d}{f}$

$\cos\theta = \frac{d}{f}$ $\qquad$ $\cos\alpha = \frac{e}{f}$

$\tan\theta = \frac{e}{d}$ $\qquad$ $\tan\alpha = \frac{d}{e}$

Enrichment

1. a.



b. Refer to the answer to question 1.a. for the measurements.

$$\text{Area A} = \frac{1}{2}bh \qquad \text{Area B} = \frac{1}{2}bh$$

$$= \frac{1}{2}(2.1)(2.1) \qquad = \frac{1}{2}(2.8)(2.8)$$

$$\doteq 2.2 \text{ cm}^2 \qquad \doteq 3.9 \text{ cm}^2$$

$$\text{Area C} = \frac{1}{2}bh$$

$$= \frac{1}{2}(3.5)(3.5)$$

$$\doteq 6.1 \text{ cm}^2$$

c. The area of the triangle on the hypotenuse is equal to the sum of the areas of the triangles on the other two sides.

2. a. $A = \pi r^2$ $\quad A = \pi r^2$ $\quad A = \pi r^2$

$= \pi(3)^2$ $\quad = \pi(4)^2$ $\quad = \pi(5)^2$

$= \pi(9)$ $\quad = \pi(16)$ $\quad = \pi(25)$

$\doteq 28.27$ cm $\quad \doteq 50.27$ $\quad \doteq 78.54$

b. The area of the circle on the hypotenuse is equal to the sum of the areas of the circles on the other two sides.

3. The areas of similar figures on the sides of right triangles are all going to conform to the same property. The areas of the figures that you draw all have the same relationship to the corresponding square on the side of the triangle (half, two-thirds, and so on).

4. Area A $= \dfrac{\pi r^2}{2}$ $\quad$ Area B $= \dfrac{\pi r^2}{2}$

$= \dfrac{\pi(6)^2}{2}$ $\quad = \dfrac{\pi(2.5)^2}{2}$

$\doteq 56.55$ cm^2 $\quad \doteq 9.82$ cm^2

Area C $= \dfrac{\pi r^2}{2}$

$= \dfrac{\pi(6.5)^2}{2}$

$\doteq 66.37$

Use the Pythagorean Theorem to find the diameter of *C*.

or

Area C = Area A + Area B

$\doteq 56.55 + 9.82$

$\doteq 66.37$ cm^2

5. $(AC)^2 = (AB)^2 - (BC)^2$

$(AC)^2 = (26)^2 - (10)^2$

$(AC)^2 = 676 - 100$

$(AC)^2 = 576$

$AC = \sqrt{576}$

$= 24$

Area X $= \frac{1}{2}bh$ $\quad$ Area Y $= \frac{1}{2}bh$

$= \frac{1}{2}(26)(52)$ $\quad = \frac{1}{2}(10)(20)$

$= 676$ cm^2 $\quad = 100$ cm^2

Area Z $= \frac{1}{2}bh$ $\quad$ or $\quad$ Area Z = Area X − Area Y

$= \frac{1}{2}(24)(48)$ $\quad = 676 - 100$

$= 576$ cm^2 $\quad = 576$ cm^2

Formula Sheet

Area

$A = s^2$	area of a square
$A = \ell wh$	area of a rectangle
$A = \frac{1}{2}bh$	area of a triangle
$A = bh$	area of a parallelogram
$A = \frac{(b_1 + b_2)h}{2}$	area of a trapezoid
$A = \frac{nsa}{2}$	area of a regular polygon
$A = \pi r^2$	area of a circle

Surface Area

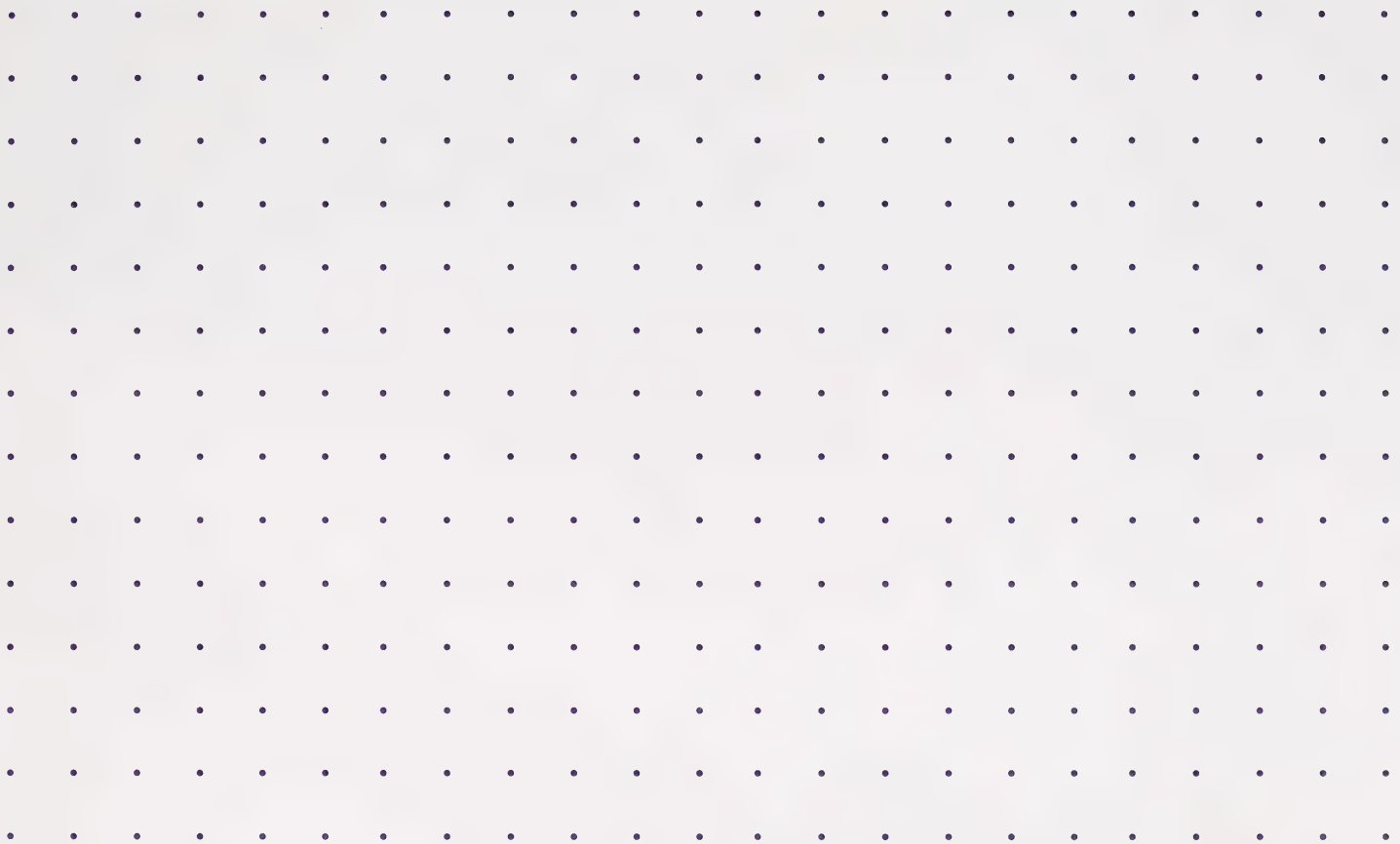
$SA = 6s^2$	surface area of a cube
$SA = 2B + PH$	surface area of a rectangular prism
$SA = 2B + CH$	surface area of a cylinder

Volume

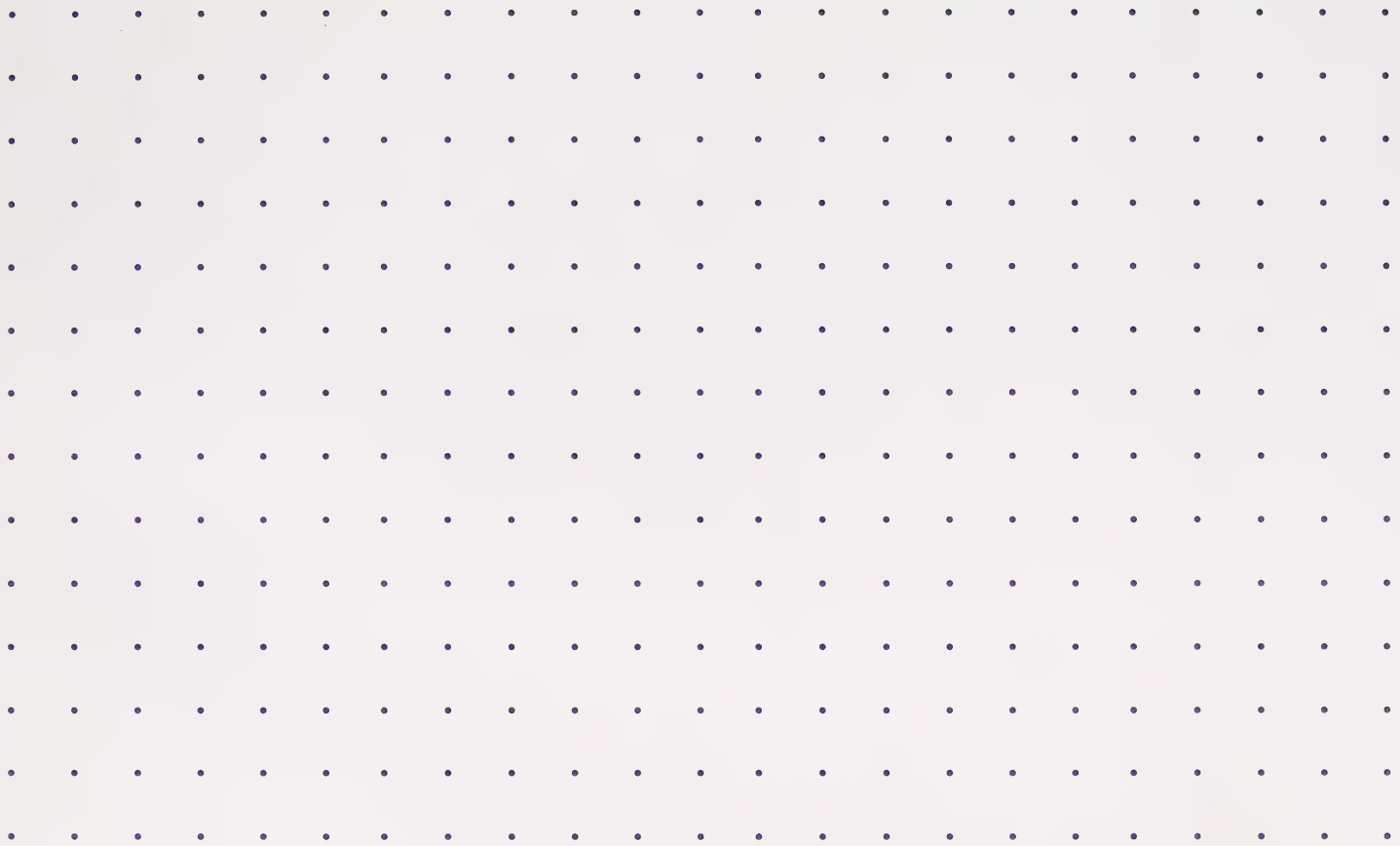
$V = BH$	volume of a prism
$V = BH$	volume of a cylinder
$V = \frac{1}{3}BH$	volume of a cone
$V = \frac{1}{3}BH$	volume of a pyramid

Dot Paper, Grids, and Cutouts

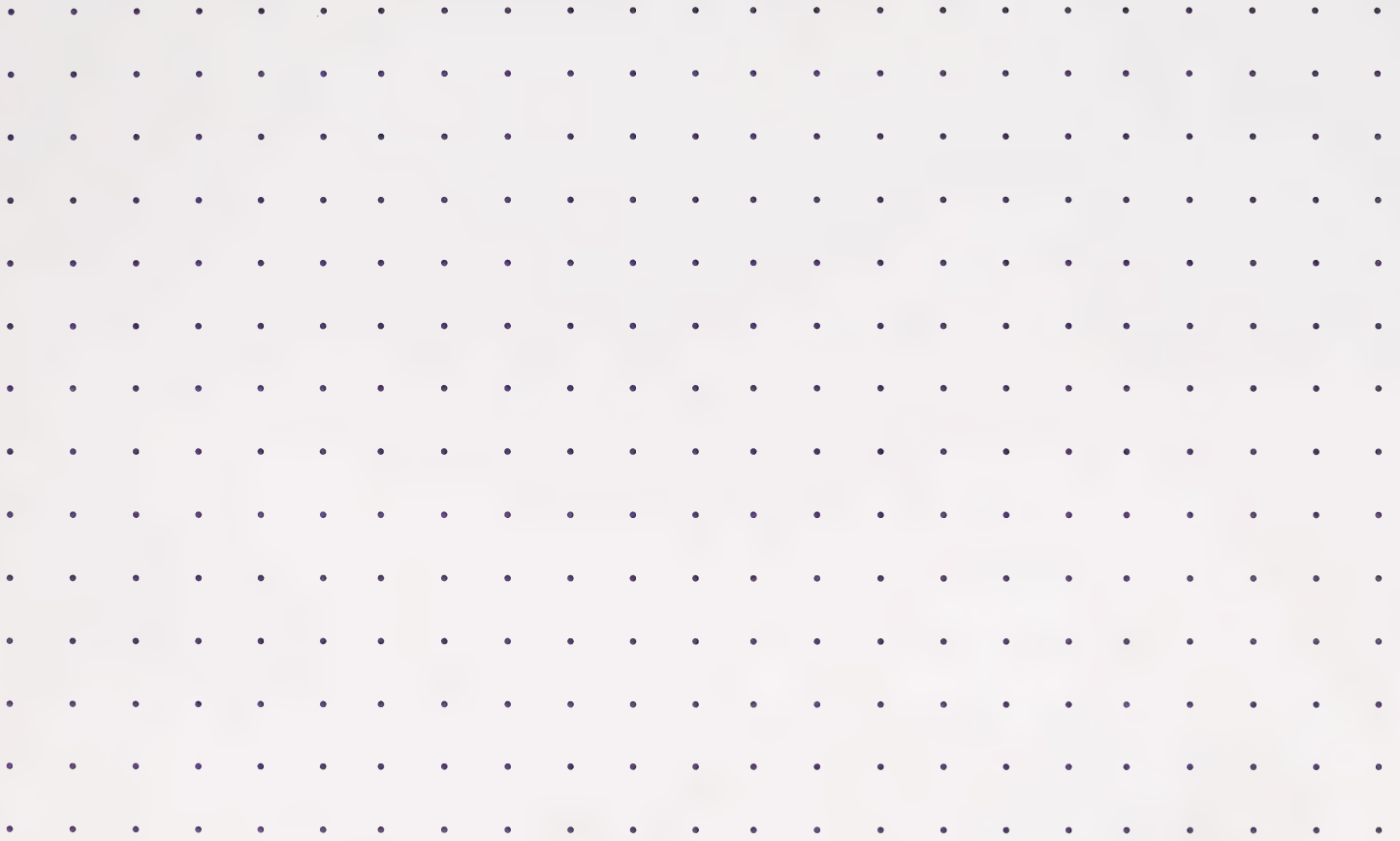
Square Dot Paper



Square Dot Paper



Square Dot Paper



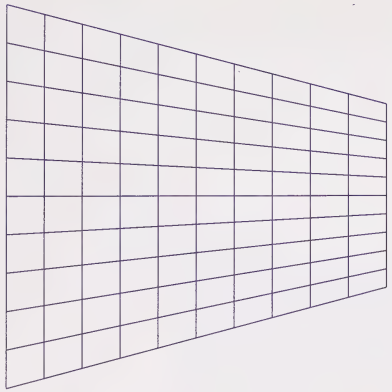
Isometric Dot Paper

Isometric Dot Paper

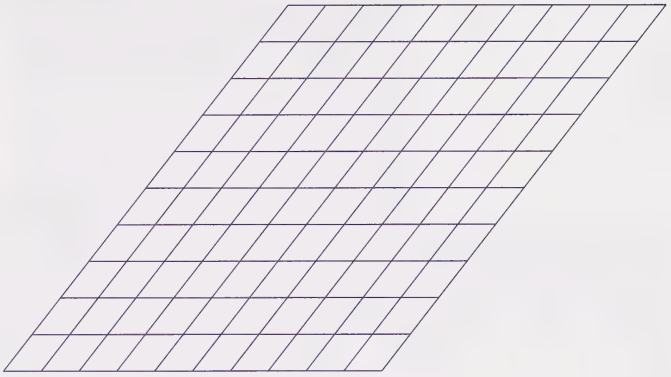
Isometric Dot Paper

Distorted Grids for Section 1: Enrichment, Question 5

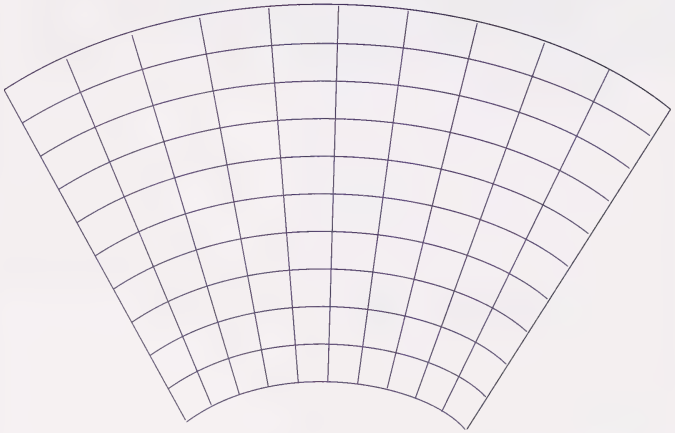
5. a.



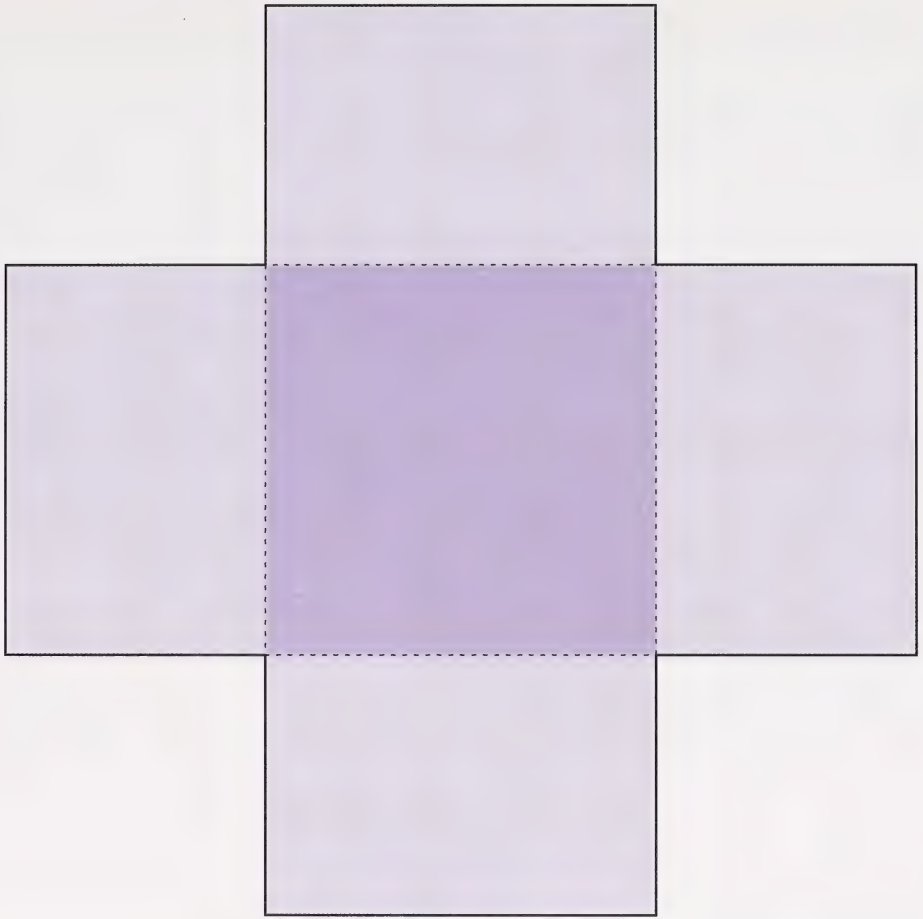
b.



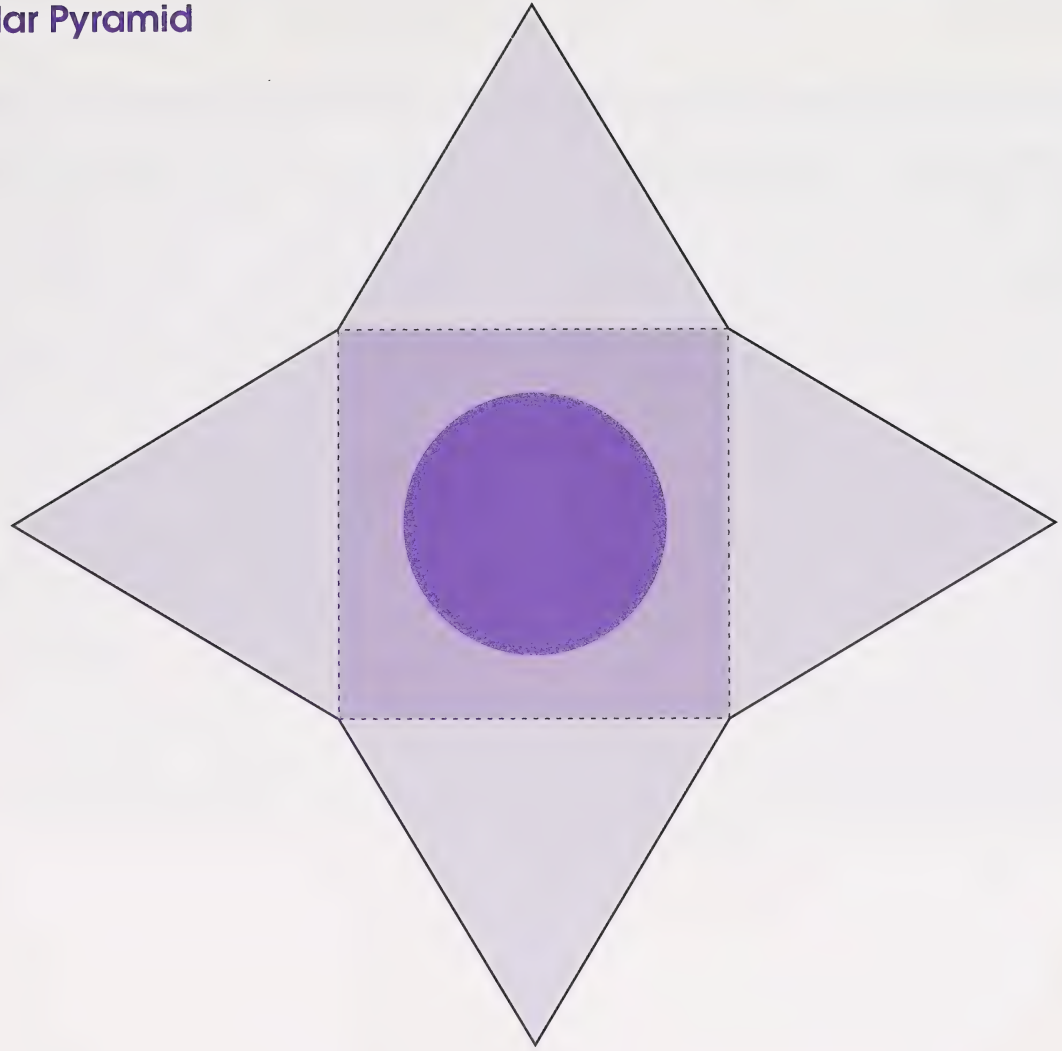
c.



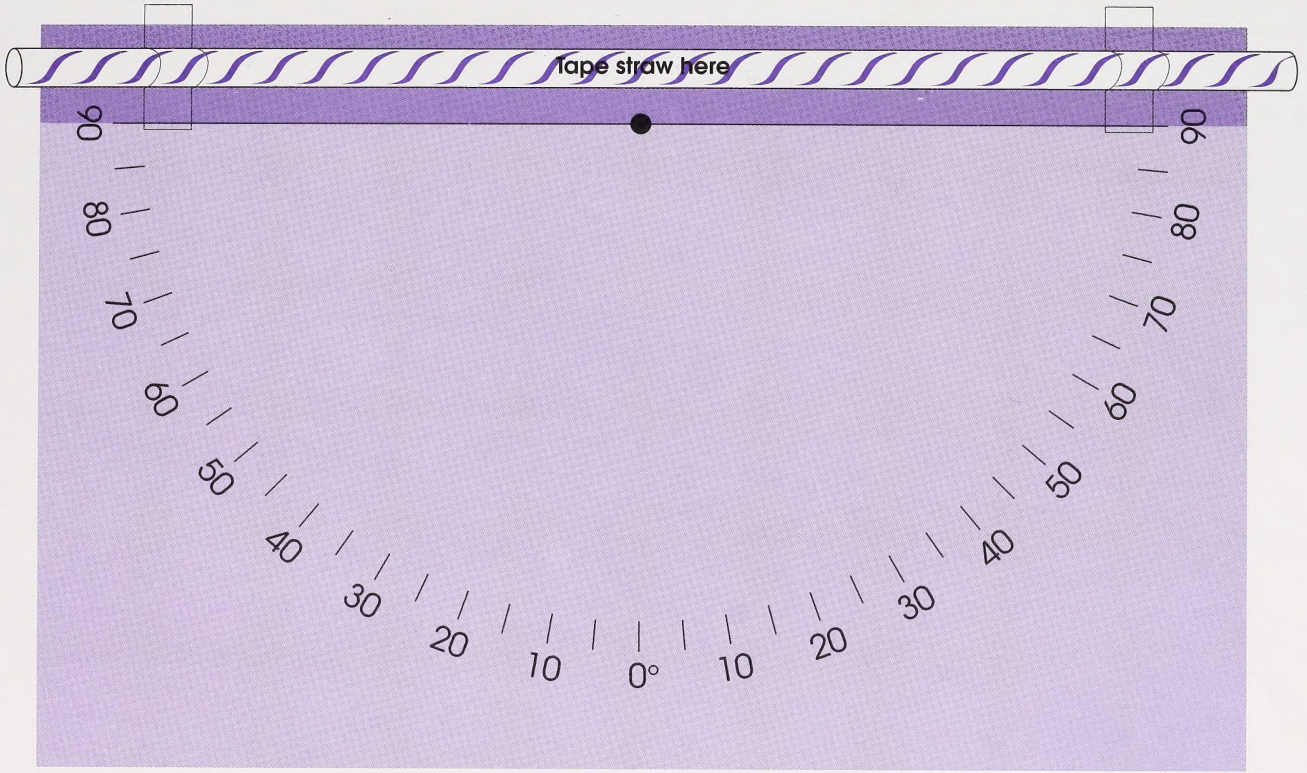
Rectangular Prism



Rectangular Pyramid



Clinometer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